



# Laplace Transforms

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May 2006



# Contents

<b>1</b>	<b>Problems</b>	<b>1</b>
1.1	Laplace Transforms . . . . .	1
1.2	Inverse Laplace Transforms . . . . .	2
1.3	Initial Value Problems . . . . .	3
1.4	Step Functions and Impulses . . . . .	4
1.5	Convolution . . . . .	5
<b>2</b>	<b>Solutions</b>	<b>7</b>
2.1	Laplace Transforms . . . . .	7
2.2	Inverse Laplace Transforms . . . . .	10
2.3	Initial Value Problems . . . . .	17
2.4	Step Functions and Impulses . . . . .	25
2.5	Convolution . . . . .	34
<b>A</b>	<b>Formulas and Properties</b>	<b>41</b>
A.1	Table of Laplace Transforms . . . . .	41
A.2	Properties of Laplace Transforms . . . . .	42
A.3	Trigonometric Identities . . . . .	43
<b>B</b>	<b>Partial Fractions</b>	<b>45</b>
B.1	Partial Fractions . . . . .	45
B.2	Cover-up Method . . . . .	46
	<b>Bibliography</b>	<b>49</b>



# Chapter 1

## Problems

### 1.1 Laplace Transforms

1. Find the Laplace transform of the following functions.

(a)  $f(t) = 4t^2 - 2t + 3$

(d)  $f(t) = e^{-2t} (4 \cos 5t + 3 \sin 5t)$

(b)  $f(t) = 3 \sin 5t - 2 \cos 3t$

(e)  $f(t) = t^3 e^{2t} + 2t e^{-t}$

(c)  $f(t) = 3e^{2t} + 5e^{-3t}$

(f)  $f(t) = (1 + e^{3t})^2$

2. Use an appropriate trigonometric identity to find the following Laplace transforms. See page 43 for a list of trigonometric identities.

(a)  $\mathcal{L}\{\cos^2 t\}$

(d)  $\mathcal{L}\{\sin(\omega t + \phi)\}$

(b)  $\mathcal{L}\{\sin 2t \cos 2t\}$

(e)  $\mathcal{L}\{\cos(\omega t + \phi)\}$

(c)  $\mathcal{L}\{\sin 3t \cos 4t\}$

(f)  $\mathcal{L}\{e^{-2t} \sin(3t + \frac{\pi}{6})\}$

3. The hyperbolic sine and hyperbolic cosine are defined by

$$\sinh t = \frac{e^t - e^{-t}}{2} \quad \text{and} \quad \cosh t = \frac{e^t + e^{-t}}{2}.$$

Evaluate both  $\mathcal{L}\{\sinh \omega t\}$  and  $\mathcal{L}\{\cosh \omega t\}$ .

4. Find both  $\mathcal{L}\{\cos \omega t\}$  and  $\mathcal{L}\{\sin \omega t\}$  by starting from Euler's formula

$$e^{j\theta} = \cos \theta + j \sin \theta$$

and

$$\mathcal{L}\{e^{at}\} = \frac{1}{s - a}.$$

5. Use the frequency differentiation property (see page 42) to derive the following formulas.

$$(a) \mathcal{L}\{t \sin \omega t\} = \frac{2\omega s}{(s^2 + \omega^2)^2} \quad (b) \mathcal{L}\{t \cos \omega t\} = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

6. Use the frequency shift property (see page 42) to find the following.

$$(a) \mathcal{L}\{t e^{2t} \sin 3t\} \quad (b) \mathcal{L}\{t e^{-3t} \cos 2t\}$$

## 1.2 Inverse Laplace Transforms

In problems 1 – 8, find the inverse Laplace transforms of the given functions.

$$\begin{array}{ll} 1. F(s) = \frac{1}{s^4} & 5. F(s) = \frac{s-2}{s^2+2s+5} \\ 2. F(s) = \frac{s+5}{s^2+4} & 6. F(s) = \frac{s+5}{(s^2+9)^2} \\ 3. F(s) = \frac{3s+1}{s^2+5} & 7. F(s) = \frac{1}{s^2-4s+7} \\ 4. F(s) = \frac{1}{(s-3)^5} & 8. F(s) = \frac{5s+2}{(s^2+6s+13)^2} \end{array}$$

In problems 9 – 16, use the method of partial fractions to find the given inverse Laplace transforms. See page 45 for a refresher on partial fractions.

$$\begin{array}{ll} 9. \mathcal{L}^{-1}\left\{\frac{s+3}{s^2+4s-5}\right\} & 13. \mathcal{L}^{-1}\left\{\frac{s^2+3s+4}{(s-2)^3}\right\} \\ 10. \mathcal{L}^{-1}\left\{\frac{s+1}{(2s-1)(s+2)}\right\} & 14. \mathcal{L}^{-1}\left\{\frac{s^2+s+4}{s^3+9s}\right\} \\ 11. \mathcal{L}^{-1}\left\{\frac{s^2+s+5}{s^4-s^2}\right\} & 15. \mathcal{L}^{-1}\left\{\frac{3s+2}{s^3(s^2+1)}\right\} \\ 12. \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+4)}\right\} & 16. \mathcal{L}^{-1}\left\{\frac{2s+1}{(s+3)(s^2+4s+13)}\right\} \end{array}$$

### 1.3 Initial Value Problems

Use Laplace transforms to solve the following initial-value problems.

1.  $y' + 4y = e^t$ ,  $y(0) = 2$
2.  $y' - y = \sin t$ ,  $y(0) = 1$
3.  $y'' + 3y' + 2y = t + 1$ ,  $y(0) = 1$ ,  $y'(0) = 0$
4.  $y'' + 4y' + 13y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 2$
5.  $y'' + 4y = 4t + 8$ ,  $y(0) = 4$ ,  $y'(0) = -1$
6.  $y'' + y' - 2y = 5e^{3t}$ ,  $y(0) = 1$ ,  $y'(0) = -4$
7.  $y'' + y' - 2y = e^t$ ,  $y(0) = 2$ ,  $y'(0) = 3$
8.  $y'' - 2y' + y = e^t$ ,  $y(0) = 3$ ,  $y'(0) = 4$
9.  $y'' + 2y' + 2y = \cos 2t$ ,  $y(0) = 0$ ,  $y'(0) = 1$
10.  $y'' + 4y = \sin 3t$ ,  $y(0) = 2$ ,  $y'(0) = 1$
11.  $y'' + \omega^2 y = \cos \omega t$ ,  $y(0) = 0$ ,  $y'(0) = 0$
12.  $y'' + 2y' + 5y = 3e^{-t} \cos 2t$ ,  $y(0) = 1$ ,  $y'(0) = 2$
13. The differential equation for a mass-spring system is

$$mx''(t) + \beta x'(t) + kx(t) = F_e(t).$$

Consider a mass-spring system with a mass  $m = 1$  kg that is attached to a spring with constant  $k = 5$  N/m. The medium offers a damping force six times the instantaneous velocity, i.e.,  $\beta = 6$  N·s/m.

- (a) Determine the position of the mass  $x(t)$  if it is released with initial conditions:  $x(0) = 3$  m,  $x'(0) = 1$  m/s. There is no external force.
  - (b) Determine the position of the mass  $x(t)$  if it released with the initial conditions:  $x(0) = 0$ ,  $x'(0) = 0$  and the system is driven by an external force  $F_e(t) = 30 \sin 2t$  in newtons with time  $t$  measured in seconds.
14. The differential equation for the current  $i(t)$  in an  $LR$  circuit is

$$L \frac{di}{dt} + Ri = V(t).$$

Find the current in an  $LR$  circuit if the initial current is  $i(0) = 0$  A given that  $L = 2$  H,  $R = 4 \Omega$ , and  $V(t) = 5e^{-t}$  volts with time  $t$  measured in seconds..

15. The differential equations for the charge  $q(t)$  in an  $LRC$  circuit is

$$Lq''(t) + Rq'(t) + \frac{q(t)}{C} = V(t).$$

Find the charge and current in an  $LRC$  circuit with  $L = 1$  H,  $R = 2 \Omega$ ,  $C = 0.25$  F and  $V = 50 \cos t$  volts if  $q(0) = 0$  C and  $i(0) = 0$  A.

## 1.4 Step Functions and Impulses

1. Find the Laplace transforms of the following functions.

$$\begin{aligned} \text{(a)} \quad f(t) &= \begin{cases} 1, & 0 \leq t < 1 \\ 3, & 1 \leq t < 2 \\ -2, & t \geq 2 \end{cases} & \text{(c)} \quad f(t) &= \begin{cases} t, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \\ \text{(b)} \quad f(t) &= \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases} & \text{(d)} \quad f(t) &= \begin{cases} 0, & 0 \leq t < 3 \\ t^2, & t \geq 3 \end{cases} \end{aligned}$$

2. Find the following inverse Laplace transforms.

$$\begin{aligned} \text{(a)} \quad \mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s+3} \right\} & & \text{(c)} \quad \mathcal{L}^{-1} \left\{ \frac{e^{-s}}{(s-1)(s+2)} \right\} \\ \text{(b)} \quad \mathcal{L}^{-1} \left\{ \frac{e^{-2\pi s}}{s^2+4} \right\} & & \text{(d)} \quad \mathcal{L}^{-1} \left\{ \frac{e^{-\pi s}}{s(s^2+1)} \right\} \end{aligned}$$

In problems 3 – 9, use Laplace transforms to solve the initial-value problem.

3.  $y' + 2y = f(t)$ ,  $y(0) = 3$ , where  $f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & t \geq 1 \end{cases}$
4.  $y'' + y = f(t)$ ,  $y(0) = 1$ ,  $y'(0) = 0$ , where  $f(t) = \begin{cases} 0, & 0 \leq t \leq \pi \\ \sin t, & t \geq \pi \end{cases}$
5.  $y'' + y' - 2y = f(t)$ ,  $y(0) = 0$ ,  $y'(0) = 0$ , where  $f(t) = \begin{cases} 1, & 0 \leq t \leq 3 \\ -1, & t \geq 3 \end{cases}$
6.  $y'' + 4y = f(t)$ ,  $y(0) = 0$ ,  $y'(0) = 2$ , where  $f(t) = \begin{cases} 1, & 0 \leq t < \pi, \\ 2, & \pi \leq t < 2\pi, \\ 0, & t \geq 2\pi \end{cases}$
7.  $y' + 3y = \delta(t - 1)$ ,  $y(0) = 2$
8.  $y'' + 4y' + 13y = \delta(t - \pi)$ ,  $y(0) = 2$ ,  $y'(0) = 1$
9.  $y'' + y = \delta(t - \frac{\pi}{2}) + 2\delta(t - \pi)$ ,  $y(0) = 3$ ,  $y'(0) = -1$
10. Find the Laplace transform of the periodic function  $f(t)$  with period  $T = 2a$  defined over one period by

$$f(t) = \begin{cases} 1, & 0 \leq t < a \\ 0, & a \leq t < 2a. \end{cases}$$



11. The differential equation for the current  $i(t)$  in an  $LR$  circuit is

$$L \frac{di}{dt} + Ri = V(t).$$

Consider an  $LR$  circuit with  $L = 1 \text{ H}$ ,  $R = 4 \Omega$  and a voltage source (in volts)

$$V(t) = \begin{cases} 2, & 0 \leq t < 1 \\ 0, & t \geq 1. \end{cases}$$

- (a) Find the current  $i(t)$  if  $i(0) = 0$ .
- (b) Compute the current at  $t = 1.5$  seconds, i.e., compute  $i(1.5)$ .
- (c) Evaluate  $\lim_{t \rightarrow \infty} i(t)$ .
- (d) Sketch the graph of the current as a function of time.

12. Consider the following function with  $a > 0$  and  $\epsilon > 0$ .

$$\delta_\epsilon(t - a) = \begin{cases} 0, & 0 \leq t < a \\ \frac{1}{\epsilon}, & a \leq t < a + \epsilon \\ 0, & t \geq a + \epsilon \end{cases}$$

- (a) Find  $\mathcal{L}\{\delta_\epsilon(t - a)\}$ .
- (b) Show that:  $\lim_{\epsilon \rightarrow 0} \mathcal{L}\{\delta_\epsilon(t - a)\} = e^{-as}$ .

## 1.5 Convolution

1. Find the following Laplace transforms.

- (a)  $\mathcal{L}\{1 * t^4\}$
- (b)  $\mathcal{L}\left\{\int_0^t \cos \theta \, d\theta\right\}$
- (c)  $\mathcal{L}\{e^t * \sin t\}$
- (d)  $\mathcal{L}\left\{\int_0^t \cos \theta \sin(t - \theta) \, d\theta\right\}$

2. Find the following inverse Laplace transforms by using convolution. Do **not** use partial fractions.

- (a)  $\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\}$
- (b)  $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-3)}\right\}$
- (c)  $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$
- (d)  $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$

3. Use convolution to derive the following formula.

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + \omega^2)^2} \right\} = \frac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}$$

4. Solve for  $f(t)$  in the following equation.

$$f(t) = t + e^{2t} + \int_0^t e^{-\theta} f(t - \theta) d\theta$$

5. Solve for  $f(t)$  in the following equation. Be sure to find *all* solutions.

$$\int_0^t f(\theta) f(t - \theta) d\theta = t^3$$

6. Solve for  $y(t)$  in the following initial-value problem.

$$y'(t) + \int_0^t e^{-2\theta} y(t - \theta) d\theta = 1, \quad y(0) = 0$$

# Chapter 2

## Solutions

### 2.1 Laplace Transforms

1. We use the table of Laplace transforms (see page 41) to answer this question.

$$(a) \mathcal{L}\{f(t)\} = 4\mathcal{L}\{t^2\} - 2\mathcal{L}\{t\} + 3\mathcal{L}\{1\} = \frac{8}{s^3} - \frac{2}{s^2} + \frac{3}{s}$$

$$(b) \mathcal{L}\{f(t)\} = 3\mathcal{L}\{\sin 5t\} - 2\mathcal{L}\{\cos 3t\} = \frac{15}{s^2 + 25} - \frac{2s}{s^2 + 9}$$

$$(c) \mathcal{L}\{f(t)\} = 3\mathcal{L}\{e^{2t}\} + 5\mathcal{L}\{e^{-3t}\} = \frac{3}{s-2} + \frac{5}{s+3}$$

$$(d) \mathcal{L}\{f(t)\} = 4\mathcal{L}\{e^{-2t} \cos 5t\} + 3\mathcal{L}\{e^{-2t} \sin 5t\} = \frac{4(s+2)}{(s+2)^2 + 25} + \frac{15}{(s+2)^2 + 25}$$

$$(e) \mathcal{L}\{f(t)\} = \mathcal{L}\{t^3 e^{2t}\} + 2\mathcal{L}\{te^{-t}\} = \frac{6}{(s-2)^4} + \frac{2}{(s+1)^2}$$

(f) Since  $f(t) = (1 + e^{3t})^2 = 1 + 2e^{3t} + e^{6t}$ , we have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} + 2\mathcal{L}\{e^{3t}\} + \mathcal{L}\{e^{6t}\} = \frac{1}{s} + \frac{2}{s-3} + \frac{1}{s-6}.$$

2. (a) Since  $\cos^2 t = \frac{1 + \cos 2t}{2}$ , we have

$$\begin{aligned} \mathcal{L}\{\cos^2 t\} &= \frac{1}{2}(\mathcal{L}\{1\} + \mathcal{L}\{\cos 2t\}) \\ &= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2 + 4}\right). \end{aligned}$$

(b) Starting from the identity  $\sin 2\theta = 2 \sin \theta \cos \theta$ , if we let  $\theta = 2t$  we get

$$\sin 4t = 2 \sin 2t \cos 2t \implies \sin 2t \cos 2t = \frac{1}{2} \sin 4t$$

and

$$\mathcal{L}\{\sin 2t \cos 2t\} = \frac{1}{2} \mathcal{L}\{\sin 4t\} = \frac{1}{2} \left( \frac{4}{s^2 + 16} \right) = \frac{2}{s^2 + 16}.$$

(c) From the identity  $\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$ , we deduce

$$\begin{aligned} \sin 3t \cos 4t &= \frac{1}{2}(\sin(3t + 4t) + \sin(3t - 4t)) \\ &= \frac{1}{2}(\sin 7t + \sin(-t)) \\ &= \frac{1}{2}(\sin 7t - \sin t) \end{aligned}$$

and

$$\mathcal{L}\{\sin 3t \cos 4t\} = \frac{1}{2} \left( \frac{7}{s^2 + 49} - \frac{1}{s^2 + 1} \right).$$

(d) From the identity  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ , we deduce

$$\sin(\omega t + \phi) = \sin \omega t \cos \phi + \cos \omega t \sin \phi$$

and

$$\begin{aligned} \mathcal{L}\{\sin(\omega t + \phi)\} &= \cos \phi \mathcal{L}\{\sin \omega t\} + \sin \phi \mathcal{L}\{\cos \omega t\} \\ &= \cos \phi \left( \frac{\omega}{s^2 + \omega^2} \right) + \sin \phi \left( \frac{s}{s^2 + \omega^2} \right). \end{aligned}$$

(e) From the identity  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ , we deduce

$$\cos(\omega t + \phi) = \cos \omega t \cos \phi - \sin \omega t \sin \phi$$

and

$$\begin{aligned} \mathcal{L}\{\cos(\omega t + \phi)\} &= \cos \phi \mathcal{L}\{\cos \omega t\} - \sin \phi \mathcal{L}\{\sin \omega t\} \\ &= \cos \phi \left( \frac{s}{s^2 + \omega^2} \right) - \sin \phi \left( \frac{\omega}{s^2 + \omega^2} \right). \end{aligned}$$

(f) Using the identity  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ , we deduce

$$\begin{aligned} e^{-2t} \sin(3t + \frac{\pi}{6}) &= e^{-2t} (\sin 3t \cos \frac{\pi}{6} + \cos 3t \sin \frac{\pi}{6}) \\ &= e^{-2t} \left( \frac{\sqrt{3}}{2} \sin 3t + \frac{1}{2} \cos 3t \right) \\ &= \frac{\sqrt{3}}{2} e^{-2t} \sin 3t + \frac{1}{2} e^{-2t} \cos 3t \end{aligned}$$

and

$$\mathcal{L}\{e^{-2t} \sin(3t + \frac{\pi}{6})\} = \frac{\sqrt{3}}{2} \left( \frac{3}{(s+2)^2 + 9} \right) + \frac{1}{2} \left( \frac{s+2}{(s+2)^2 + 9} \right).$$

3. Using  $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ , we deduce

$$\begin{aligned}\mathcal{L}\{\sinh \omega t\} &= \frac{1}{2}(\mathcal{L}\{e^{\omega t}\} - \mathcal{L}\{e^{-\omega t}\}) \\ &= \frac{1}{2}\left(\frac{1}{s-\omega} - \frac{1}{s+\omega}\right) \\ &= \frac{1}{2}\left(\frac{s+\omega - (s-\omega)}{s^2 - \omega^2}\right) \\ &= \frac{\omega}{s^2 - \omega^2}\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}\{\cosh \omega t\} &= \frac{1}{2}(\mathcal{L}\{e^{\omega t}\} + \mathcal{L}\{e^{-\omega t}\}) \\ &= \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right) \\ &= \frac{1}{2}\left(\frac{s+\omega + s-\omega}{s^2 - \omega^2}\right) \\ &= \frac{s}{s^2 - \omega^2}.\end{aligned}$$

4. From Euler's formula, we get

$$\mathcal{L}\{e^{j\omega t}\} = \mathcal{L}\{\cos \omega t\} + j\mathcal{L}\{\sin \omega t\}. \quad (2.1)$$

From  $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ , we deduce

$$\mathcal{L}\{e^{j\omega t}\} = \frac{1}{s-j\omega} = \frac{1}{s-j\omega} \left(\frac{s+j\omega}{s+j\omega}\right) = \frac{s+j\omega}{s^2 + \omega^2}. \quad (2.2)$$

Since the real and imaginary parts in equations (2.1) and (2.2) are equal, we conclude that

$$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2} \quad \text{and} \quad \mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}.$$

5. We will use the property  $\mathcal{L}\{tf(t)\} = -F'(s)$ .

(a) If  $f(t) = \sin \omega t$ , we have  $F(s) = \frac{\omega}{s^2 + \omega^2}$ . Therefore

$$\begin{aligned}\mathcal{L}\{t \sin \omega t\} &= -F'(s) = -\frac{d}{ds} \left(\frac{\omega}{s^2 + \omega^2}\right) \\ &= \frac{2\omega s}{(s^2 + \omega^2)^2}.\end{aligned}$$

(b) If  $f(t) = \cos \omega t$ , we have  $F(s) = \frac{s}{s^2 + \omega^2}$ . Therefore

$$\begin{aligned}\mathcal{L}\{t \cos \omega t\} &= -F'(s) = -\frac{d}{ds} \left( \frac{s}{s^2 + \omega^2} \right) \\ &= \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}.\end{aligned}$$

6. We use the frequency shift property  $\mathcal{L}\{e^{at} f(t)\} = F(s - a)$ .

(a) Since  $\mathcal{L}\{t \sin 3t\} = \frac{6s}{(s^2 + 9)^2}$ , we conclude that

$$\mathcal{L}\{te^{2t} \sin 3t\} = \frac{6(s-2)}{((s-2)^2 + 9)^2} = \frac{6(s-2)}{(s^2 - 4s + 13)^2}.$$

(b) Since  $\mathcal{L}\{t \cos 2t\} = \frac{s^2 - 4}{(s^2 + 4)^2}$ , we conclude that

$$\mathcal{L}\{te^{-3t} \cos 2t\} = \frac{(s+3)^2 - 4}{((s+3)^2 + 4)^2} = \frac{s^2 + 6s + 5}{(s^2 + 6s + 13)^2}.$$

## 2.2 Inverse Laplace Transforms

$$1. \mathcal{L}^{-1} \left\{ \frac{1}{s^4} \right\} = \frac{1}{3!} \mathcal{L}^{-1} \left\{ \frac{3!}{s^4} \right\} = \boxed{\frac{t^3}{6}}$$

$$2. \mathcal{L}^{-1} \left\{ \frac{s+5}{s^2+4} \right\} = \mathcal{L}^{-1} \left\{ \frac{s}{s^2+4} \right\} + \frac{5}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^2+4} \right\} = \boxed{\cos 2t + \frac{5}{2} \sin 2t}$$

$$3. \mathcal{L}^{-1} \left\{ \frac{3s+1}{s^2+5} \right\} = 3 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+5} \right\} + \frac{1}{\sqrt{5}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{5}}{s^2+5} \right\}$$

$$\boxed{\mathcal{L}^{-1} \left\{ \frac{3s+1}{s^2+5} \right\} = 3 \cos \sqrt{5}t + \frac{1}{\sqrt{5}} \sin \sqrt{5}t}$$

$$4. \mathcal{L}^{-1} \left\{ \frac{1}{(s-3)^5} \right\} = \frac{1}{4!} \mathcal{L}^{-1} \left\{ \frac{4!}{(s-3)^5} \right\} = \boxed{\frac{t^4 e^{3t}}{24}}$$

5. By completing the square, we get

$$s^2 + 2s + 5 = (s+1)^2 + 4.$$

Observe that

$$\frac{s-2}{s^2+2s+5} = \frac{(s+1)-3}{(s+1)^2+4}.$$

Therefore,

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s-2}{s^2+2s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} \\ &= \boxed{e^{-t}\cos 2t - \frac{3}{2}e^{-t}\sin 2t.} \end{aligned}$$

$$6. \quad \mathcal{L}^{-1}\left\{\frac{s+5}{(s^2+9)^2}\right\} = \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{6s}{(s^2+9)^2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{(s^2+9)^2}\right\}$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s+5}{(s^2+9)^2}\right\} &= \frac{1}{6}t\sin 3t + 5\left(\frac{\sin 3t - 3t\cos 3t}{2(3^3)}\right) \\ &= \boxed{\frac{1}{6}t\sin 3t + \frac{5}{54}\sin 3t - \frac{5}{18}t\cos 3t.} \end{aligned}$$

7. By completing the square, we get

$$s^2 - 4s + 7 = (s-2)^2 + 3.$$

Therefore,

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+7}\right\} &= \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s-2)^2+3}\right\} \\ &= \boxed{\frac{1}{\sqrt{3}}e^{2t}\sin \sqrt{3}t.} \end{aligned}$$

8. First we complete the square to get

$$s^2 + 6s + 13 = (s+3)^2 + 4.$$

Observe that

$$\frac{5s+2}{(s^2+6s+13)^2} = \frac{5(s+3)-13}{((s+3)^2+4)^2}.$$

We now use the frequency shift theorem  $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$  to conclude that

$$\mathcal{L}^{-1}\left\{\frac{5(s+3)-13}{((s+3)^2+4)^2}\right\} = e^{-3t}\mathcal{L}^{-1}\left\{\frac{5s-13}{(s^2+4)^2}\right\}.$$

Now let's focus on  $\mathcal{L}^{-1} \left\{ \frac{5s - 13}{(s^2 + 4)^2} \right\}$ .

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{5s - 13}{(s^2 + 4)^2} \right\} &= \frac{5}{4} \mathcal{L}^{-1} \left\{ \frac{4s}{(s^2 + 4)^2} \right\} - 13 \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + 4)^2} \right\} \\ &= \frac{5}{4} t \sin 2t - 13 \left( \frac{\sin 2t - 2t \cos 2t}{2(2)^3} \right) \\ &= \frac{5}{4} t \sin 2t - \frac{13}{16} \sin 2t + \frac{13}{8} t \cos 2t \end{aligned}$$

Therefore,

$$\mathcal{L}^{-1} \left\{ \frac{5s + 2}{(s^2 + 6s + 13)^2} \right\} = e^{-3t} \left( \frac{5}{4} t \sin 2t - \frac{13}{16} \sin 2t + \frac{13}{8} t \cos 2t \right).$$

9. We can factor the denominator to obtain

$$s^2 + 4s - 5 = (s - 1)(s + 5).$$

The partial fractions decomposition is of the form

$$\frac{s + 3}{(s - 1)(s + 5)} = \frac{A}{s - 1} + \frac{B}{s + 5}.$$

Using the cover-up method we obtain  $A$  and  $B$  as follows.

$$A = \frac{s + 3}{\cancel{(s - 1)}(s + 5)} \Big|_{s=1} = \frac{2}{3} \quad \text{and} \quad B = \frac{s + 3}{(s - 1)\cancel{(s + 5)}} \Big|_{s=-5} = \frac{1}{3}$$

Then,

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{s + 3}{(s - 1)(s + 5)} \right\} &= \frac{2}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s - 1} \right\} + \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s + 5} \right\} \\ &= \boxed{\frac{2}{3} e^t + \frac{1}{3} e^{-5t}}. \end{aligned}$$

10. The partial fractions decomposition is of the form

$$\frac{s + 1}{(2s - 1)(s + 2)} = \frac{A}{2s - 1} + \frac{B}{s + 2}.$$

Using the cover-up method we obtain  $A$  and  $B$  as follows.

$$A = \frac{s + 1}{\cancel{(2s - 1)}(s + 2)} \Big|_{s=1/2} = \frac{3}{5} \quad \text{and} \quad B = \frac{s + 1}{(2s - 1)\cancel{(s + 2)}} \Big|_{s=-2} = \frac{1}{5}$$



Then,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+1}{(2s-1)(s+2)}\right\} &= \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{2s-1}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= \frac{3}{5 \cdot 2}\mathcal{L}^{-1}\left\{\frac{1}{s-1/2}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= \boxed{\frac{3}{10}e^{t/2} + \frac{1}{5}e^{-2t}}.\end{aligned}$$

11. We start by factoring the denominator to get  $s^4 - s^2 = s^2(s-1)(s+1)$ . The partial fractions decomposition is of the form

$$\frac{s^2 + s + 5}{s^2(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}.$$

By multiplying both sides by  $s^2(s-1)(s+1)$  we get

$$s^2 + s + 5 = As(s-1)(s+1) + B(s-1)(s+1) + Cs^2(s+1) + Ds^2(s-1).$$

To find the constants, let's assign values to  $s$  as follows.

$$s = 0 \implies 5 = 0 - B + 0 + 0 \implies B = -5$$

$$s = 1 \implies 7 = 0 + 0 + 2C + 0 \implies C = 7/2$$

$$s = -1 \implies 5 = 0 + 0 + 0 - 2D \implies D = -5/2$$

$$s = 2 \implies 11 = 6A + 3B + 12C + 4D \implies A = \frac{11 - 3B - -12C - 4D}{6} = -1$$

Then,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^2 + s + 5}{s^4 - s^2}\right\} &= -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 5\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= \boxed{-1 - 5t + \frac{7}{2}e^t - \frac{5}{2}e^{-t}}.\end{aligned}$$

12. The partial fractions decomposition is of the form

$$\frac{1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+c}{s^2+4}.$$

We can use the cover-up method to find  $A$ ,  $B$ ,  $C$  as follows.

$$A = \frac{1}{\cancel{(s+1)}(s^2+4)} \Big|_{s=-1} = \frac{1}{5}$$

$$\begin{aligned} (Bs + C)|_{s=2j} &= \frac{1}{(s+1)(s^2+4)} \Big|_{s=2j} \\ 2Bj + C &= \frac{1}{1+2j} \\ &= \frac{1}{1+2j} \left( \frac{1-2j}{1-2j} \right) \\ &= \frac{1-2j}{5}. \end{aligned}$$

Since the real and imaginary parts are equal on both sides, we get

$$B = -\frac{1}{5} \quad \text{and} \quad C = \frac{1}{5}.$$

Then,

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)(s^2+4)} \right\} &= \frac{1}{5} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} - \frac{1}{5} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+4} \right\} + \frac{1}{5 \cdot 2} \mathcal{L}^{-1} \left\{ \frac{2}{s^2+4} \right\} \\ &= \boxed{\frac{1}{5} e^{-t} - \frac{1}{5} \cos 2t + \frac{1}{10} \sin 2t}. \end{aligned}$$

**13.** The partial fractions decomposition is of the form

$$\frac{s^2 + 3s + 4}{(s-2)^3} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{(s-2)^3}.$$

By multiplying both sides by  $(s-2)^3$  and expanding, we get

$$\begin{aligned} s^2 + 3s + 4 &= A(s-2)^2 + B(s-2) + C \\ &= A(s^2 - 4s + 4) + Bs - 2B + C \\ &= As^2 + (B - 4A)s + (C - 2B + 4A). \end{aligned}$$

We conclude that

$$A = 1, \quad B - 4A = 3 \implies B = 7, \quad C - 2B + 4A = 4 \implies C = 14.$$

Then,

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{s^2 + 3s + 4}{(s-2)^3} \right\} &= \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} + 7 \mathcal{L}^{-1} \left\{ \frac{1}{(s-2)^2} \right\} + \frac{14}{2!} \mathcal{L}^{-1} \left\{ \frac{2!}{(s-2)^3} \right\} \\ &= \boxed{e^{2t} + 7te^{2t} + 7t^2e^{2t}}. \end{aligned}$$

**14.** We factor the denominator to get  $s^3 + 9s = s(s^2 + 9)$ . The partial fractions decomposition is of the form

$$\frac{s^2 + s + 4}{s(s^2 + 9)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 9}.$$

By multiplying both sides by  $s(s^2 + 9)$  and expanding, we get

$$\begin{aligned} s^2 + s + 4 &= A(s^2 + 9) + (Bs + C)s \\ &= As^2 + 9A + Bs^2 + Cs \\ &= (A + B)s^2 + Cs + 9A. \end{aligned}$$

We conclude that

$$9A = 4 \implies A = 4/9, \quad C = 1, \quad A + B = 1 \implies B = 5/9.$$

Then,

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{s^2 + s + 4}{s^3 + 9s} \right\} &= \frac{4}{9} \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} + \frac{5}{9} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 9} \right\} + \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{3}{s^2 + 9} \right\} \\ &= \boxed{\frac{4}{9} + \frac{5}{9} \cos 3t + \frac{1}{3} \sin 3t.} \end{aligned}$$

**15.** The partial fractions decomposition is of the form

$$\frac{3s + 2}{s^3(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2 + 1}.$$

By multiplying both sides by  $s^3(s^2 + 1)$  and expanding, we get

$$\begin{aligned} 3s + 2 &= As^2(s^2 + 1) + Bs(s^2 + 1) + C(s^2 + 1) + (Ds + E)s^3 \\ &= As^4 + As^2 + Bs^3 + Bs + Cs^2 + C + Ds^4 + Es^3 \\ 0s^4 + 0s^3 + 0s^2 + 3s + 2 &= (A + D)s^4 + (B + E)s^3 + (A + C)s^2 + Bs + C. \end{aligned}$$

We conclude that

$$\begin{aligned} C &= 2 \\ B &= 3 \\ A + C &= 0 \implies A = -2 \\ B + E &= 0 \implies E = -3 \\ A + D &= 0 \implies D = 2. \end{aligned}$$

Then,

$$\mathcal{L}^{-1} \left\{ \frac{3s + 2}{s^3(s^2 + 1)} \right\} = -2 + 3t + t^2 + 2 \cos t - 3 \sin t.$$

**16.** First we complete the square to get  $s^2 + 4s + 13 = (s + 2)^2 + 9$ . We can look for a partial fractions decomposition in the form

$$\frac{2s + 1}{(s + 3)(s^2 + 4s + 13)} = \frac{A}{s + 3} + \frac{B(s + 2) + C}{(s + 2)^2 + 9}.$$

**Method 1.** Let's multiply both sides by  $(s+3)((s+2)^2+9)$  to get

$$2s+1 = A((s+2)^2+9) + (B(s+2)+C)(s+3).$$

To find the constants  $A$ ,  $B$ , and  $C$ , let's assign values to  $s$  as follows.

$$s = -3 \implies -5 = 10A + 0 \implies A = -1/2$$

$$s = -2 \implies -3 = 9A + C \implies C = -3 - 9A = 3/2$$

$$s = -1 \implies -1 = 10A + 2B + 2C \implies B = (-3 - 10A - 2C)/2 = 1/2$$

**Method 2.** We can use the cover-up method to find  $A$ ,  $B$ , and  $C$ .

$$A = \frac{2s+1}{\cancel{(s+3)}(s^2+4s+13)} \Big|_{s=-3} = -\frac{1}{2}.$$

Since  $s^2+4s+13 = (s+2)^2+9 = 0$  if  $s = -2+3j$ , then

$$\begin{aligned} (B(s+2)+C)|_{s=-2+3j} &= \frac{2s+1}{(s+3)\cancel{(s^2+4s+13)}} \Big|_{s=-2+3j} \\ 3Bj+C &= \frac{2(-2+3j)+1}{(-2+3j)+3} \\ &= \frac{-3+6j}{1+3j} \\ &= \left(\frac{-3+6j}{1+3j}\right) \left(\frac{1-3j}{1-3j}\right) \\ 3Bj+C &= \frac{15j+15}{10}. \end{aligned}$$

Since the real and imaginary parts are equal on both sides, we get

$$3B = \frac{15}{10} \implies B = \frac{1}{2} \quad \text{and} \quad C = \frac{15}{10} = \frac{3}{2}.$$

Then,

$$\frac{2s+1}{(s+3)(s^2+4s+13)} = -\frac{1}{2} \left(\frac{1}{s+3}\right) + \frac{1}{2} \left(\frac{s+2}{(s+2)^2+9}\right) + \frac{1}{2} \left(\frac{3}{(s+2)^2+9}\right)$$

and

$$\mathcal{L}^{-1} \left\{ \frac{2s+1}{(s+3)(s^2+4s+13)} \right\} = -\frac{1}{2} e^{3t} + \frac{1}{2} e^{-2t} (\cos 3t + \sin 3t).$$

## 2.3 Initial Value Problems

1. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y'(t)\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{e^t\}$$

Since  $y(0) = 2$ , we get

$$(sY(s) - 2) + 4Y(s) = \frac{1}{s-1}.$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{2}{s+4} + \frac{1}{(s+4)(s-1)} = \frac{2s-1}{(s+4)(s-1)}.$$

Using partial fractions, we obtain

$$Y(s) = \frac{9}{5} \left( \frac{1}{s+4} \right) + \frac{1}{5} \left( \frac{1}{s-1} \right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = \frac{9}{5} e^{-4t} + \frac{1}{5} e^t}$$

2. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y'(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{\sin t\}$$

Since  $y(0) = 1$ , we get

$$(sY(s) - 1) - Y(s) = \frac{1}{s^2 + 1}.$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{1}{s-1} + \frac{1}{(s-1)(s^2+1)} = \frac{s^2+2}{(s-1)(s^2+1)}.$$

Using partial fractions, we obtain

$$Y(s) = \frac{3}{2} \left( \frac{1}{s-1} \right) - \frac{1}{2} \left( \frac{s}{s^2+1} \right) - \frac{1}{2} \left( \frac{1}{s^2+1} \right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = \frac{3}{2} e^t - \frac{1}{2} \cos t - \frac{1}{2} \sin t}$$

3. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y''(t)\} + 3\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{1\}$$

Since  $y(0) = 1$  and  $y'(0) = 0$ , we get

$$(s^2Y(s) - s - 0) + 3(sY(s) - 1) + 2Y(s) = \frac{1}{s^2} + \frac{1}{s}.$$

Solving for  $Y(s)$  and simplifying gives

$$\begin{aligned} Y(s) &= \frac{s+3}{s^2+3s+2} + \frac{1}{s^2(s^2+3s+2)} + \frac{1}{s(s^2+3s+2)} \\ &= \frac{s^3+3s^2+s+1}{s^2(s+1)(s+2)}. \end{aligned}$$

Using partial fractions, we obtain

$$Y(s) = \frac{1}{2} \left( \frac{1}{s^2} \right) - \frac{1}{4} \left( \frac{1}{s} \right) + 2 \left( \frac{1}{s+1} \right) - \frac{3}{4} \left( \frac{1}{s+2} \right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = \frac{1}{2}t - \frac{1}{4} + 2e^{-t} - \frac{3}{4}e^{-2t}}$$

4. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y'(t)\} + 13\mathcal{L}\{y(t)\} = 0$$

Since  $y(0) = 1$  and  $y'(0) = 2$ , we get

$$(s^2Y(s) - s - 2) + 4(sY(s) - 1) + 13Y(s) = 0.$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{s+6}{s^2+4s+13}.$$

By completing the square of the denominator, we get

$$s^2 + 4s + 13 = (s+2)^2 + 9.$$

Therefore,

$$Y(s) = \frac{(s+2)+4}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+9} + \frac{4}{3} \left( \frac{3}{(s+2)^2+9} \right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = e^{-2t} \cos 3t + \frac{4}{3} e^{-2t} \sin 3t}$$

5. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = 4\mathcal{L}\{t\} + \mathcal{L}\{8\}$$

Since  $y(0) = 4$  and  $y'(0) = -1$ , we get

$$(s^2Y(s) - 4s + 1) + 4Y(s) = \frac{4}{s^2} + \frac{8}{s}.$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{4s - 1}{s^2 + 4} + \frac{4}{s^2(s^2 + 4)} + \frac{8}{s(s^2 + 4)} = \frac{4s^3 - s^2 + 8s + 4}{s^2(s^2 + 4)}.$$

Using partial fractions, we obtain

$$Y(s) = \frac{1}{s^2} + 2\left(\frac{1}{s}\right) + 2\left(\frac{s}{s^2 + 4}\right) - \frac{2}{s^2 + 4}.$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = t + 2 + \cos 2t - \sin 2t}$$

6. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y''(t)\} + \mathcal{L}\{y'(t)\} - 2\mathcal{L}\{y(t)\} = 5\mathcal{L}\{e^{3t}\}$$

Since  $y(0) = 1$  and  $y'(0) = -4$ , we get

$$(s^2Y(s) - s + 4) + (sY(s) - 1) - 2Y(s) = \frac{5}{s - 3}.$$

Solving for  $Y(s)$  and simplifying gives

$$\begin{aligned} Y(s) &= \frac{s - 3}{s^2 + s - 2} + \frac{5}{(s^2 + s - 2)(s - 3)} \\ &= \frac{(s - 3)^2 + 5}{(s^2 + s - 2)(s - 3)} \\ &= \frac{s^2 - 6s + 14}{(s + 2)(s - 1)(s - 3)}. \end{aligned}$$

Using partial fractions, we obtain

$$Y(s) = 2\left(\frac{1}{s + 2}\right) - \frac{3}{2}\left(\frac{1}{s - 1}\right) + \frac{1}{2}\left(\frac{1}{s - 3}\right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = 2e^{-2t} - \frac{3}{2}e^t + \frac{1}{2}e^{3t}}$$

7. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y''(t)\} + \mathcal{L}\{y'(t)\} - 2\mathcal{L}\{y(t)\} = \mathcal{L}\{e^t\}$$

Since  $y(0) = 2$  and  $y'(0) = 3$ , we get

$$(s^2Y(s) - 2s - 3) + (sY(s) - 2) - 2Y(s) = \frac{1}{s-1}.$$

Solving for  $Y(s)$  and simplifying gives

$$\begin{aligned} Y(s) &= \frac{2s+5}{s^2+s-2} + \frac{1}{(s^2+s-2)(s-1)} \\ &= \frac{(2s+5)(s-1)+1}{(s^2+s-2)(s-1)} \\ &= \frac{2s^2+3s-4}{(s+2)(s-1)^2}. \end{aligned}$$

Using partial fractions, we obtain

$$Y(s) = -\frac{2}{9} \left( \frac{1}{s+2} \right) + \frac{20}{9} \left( \frac{1}{s-1} \right) + \frac{1}{3} \left( \frac{1}{(s-1)^2} \right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = -\frac{2}{9}e^{-2t} + \frac{20}{9}e^t + \frac{1}{3}te^t}$$

8. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y''(t)\} - 2\mathcal{L}\{y'(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{e^t\}$$

Since  $y(0) = 3$  and  $y'(0) = 4$ , we get

$$(s^2Y(s) - 3s - 4) - 2(sY(s) - 3) + Y(s) = \frac{1}{s-1}.$$

Solving for  $Y(s)$  and simplifying gives

$$\begin{aligned} Y(s) &= \frac{3s-2}{s^2-2s+1} + \frac{1}{(s^2-2s+1)(s-1)} \\ &= \frac{3(s-1)+1}{(s-1)^2} + \frac{1}{(s-1)^3} \\ &= 3 \left( \frac{1}{s-1} \right) + \frac{1}{(s-1)^2} + \frac{1}{2!} \left( \frac{2!}{(s-1)^3} \right). \end{aligned}$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = 3e^t + te^t + \frac{1}{2}t^2e^t}$$



9. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y''(t)\} + 2\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} = \mathcal{L}\{\cos 2t\}$$

Since  $y(0) = 0$  and  $y'(0) = 1$ , we get

$$(s^2Y(s) - 1) + 2sY(s) + 2Y(s) = \frac{s}{s^2 + 4}.$$

Solving for  $Y(s)$  and simplifying gives

$$\begin{aligned} Y(s) &= \frac{1}{s^2 + 2s + 2} + \frac{s}{(s^2 + 2s + 2)(s^2 + 4)} \\ &= \frac{s^2 + s + 4}{(s^2 + 2s + 2)(s^2 + 4)} \end{aligned}$$

Using partial fractions, we obtain

$$Y(s) = \frac{1}{10} \left( \frac{s + 8}{s^2 + 2s + 2} \right) - \frac{1}{10} \left( \frac{s - 4}{s^2 + 4} \right)$$

By completing the square, we get

$$s^2 + 2s + 2 = (s + 1)^2 + 1.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{1}{10} \left( \frac{(s + 1) + 7}{(s + 1)^2 + 1} \right) - \frac{1}{10} \left( \frac{s - 4}{s^2 + 4} \right) \\ &= \frac{1}{10} \left( \frac{s + 1}{(s + 1)^2 + 1} \right) + \frac{7}{10} \left( \frac{1}{(s + 1)^2 + 1} \right) - \frac{1}{10} \left( \frac{s}{s^2 + 4} \right) + \frac{4}{10 \cdot 2} \left( \frac{2}{s^2 + 4} \right) \end{aligned}$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = \frac{1}{10} e^{-t} \cos t + \frac{7}{10} e^{-t} \sin t - \frac{1}{10} \cos 2t + \frac{1}{5} \sin 2t}$$

10. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{\sin 3t\}$$

Since  $y(0) = 2$  and  $y'(0) = 1$ , we get

$$(s^2Y(s) - 2s - 1) + 4Y(s) = \frac{3}{s^2 + 9}.$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{2s + 1}{s^2 + 4} + \frac{3}{(s^2 + 4)(s^2 + 9)}.$$

Using partial fractions, we obtain

$$\frac{3}{(s^2 + 4)(s^2 + 9)} = \frac{3}{5(s^2 + 4)} - \frac{3}{5(s^2 + 9)}.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{2s + 1}{s^2 + 4} + \frac{3}{5(s^2 + 4)} - \frac{3}{5(s^2 + 9)} \\ &= 2 \left( \frac{s}{s^2 + 4} \right) + \frac{4}{5} \left( \frac{2}{(s^2 + 4)} \right) - \frac{1}{5} \left( \frac{3}{s^2 + 9} \right). \end{aligned}$$

Taking the inverse Laplace transform gives the answer.

$$y(t) = 2 \cos 2t + \frac{4}{5} \sin 2t - \frac{1}{5} \sin 3t$$

11. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y''(t)\} + \omega^2 \mathcal{L}\{y(t)\} = \mathcal{L}\{\cos \omega t\}$$

Since  $y(0) = 0$  and  $y'(0) = 0$ , we get

$$s^2 Y(s) + \omega^2 Y(s) = \frac{s}{s^2 + \omega^2}.$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{s}{(s^2 + \omega^2)^2} = \frac{1}{2\omega} \left( \frac{2\omega s}{(s^2 + \omega^2)^2} \right).$$

Taking the inverse Laplace transform gives the answer.

$$y(t) = \frac{1}{2\omega} t \sin \omega t$$

12. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y''(t)\} + 2\mathcal{L}\{y'(t)\} + 5\mathcal{L}\{y(t)\} = 3\mathcal{L}\{e^{-t} \cos 2t\}$$

Since  $y(0) = 1$  and  $y'(0) = 2$ , we get

$$(s^2 Y(s) - s - 2) + 2(sY(s) - 1) + 5Y(s) = \frac{3(s + 1)}{(s + 1)^2 + 4}.$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{s + 4}{s^2 + 2s + 5} + \frac{3(s + 1)}{(s^2 + 2s + 5)((s + 1)^2 + 4)}.$$

Since  $s^2 + 2s + 5 = (s + 1)^2 + 4$ , we obtain

$$\begin{aligned} Y(s) &= \frac{(s+1)+3}{(s+1)^2+4} + \frac{3(s+1)}{((s+1)^2+4)^2} \\ &= \frac{s+1}{(s+1)^2+4} + \frac{3}{2} \left( \frac{2}{(s+1)^2+4} \right) + \frac{3(s+1)}{((s+1)^2+4)^2}. \end{aligned}$$

Taking the inverse Laplace transform and using frequency shift property

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} \mathcal{L}^{-1}\{F(s)\}$$

we get

$$\begin{aligned} y(t) &= e^{-t} \cos(2t) + \frac{3}{2} e^{-t} \sin(2t) + \mathcal{L}^{-1} \left\{ \frac{3(s+1)}{((s+1)^2+4)^2} \right\} \\ &= e^{-t} \cos(2t) + \frac{3}{2} e^{-t} \sin(2t) + e^{-t} \mathcal{L}^{-1} \left\{ \frac{3s}{(s^2+4)^2} \right\} \\ &= e^{-t} \cos(2t) + \frac{3}{2} e^{-t} \sin(2t) + \frac{3}{4} e^{-t} \mathcal{L}^{-1} \left\{ \frac{4s}{(s^2+4)^2} \right\}. \end{aligned}$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = e^{-t} \cos(2t) + \frac{3}{2} e^{-t} \sin(2t) + \frac{3}{4} t e^{-t} \sin(2t)}$$

**13.** The differential equation of the mass-spring system is

$$x''(t) + 6x'(t) + 5x(t) = F_e(t).$$

(a) We have  $F_e(t) = 0$  and the initial conditions are

$$x(0) = 3 \quad \text{and} \quad x'(0) = 1.$$

Take the Laplace transform on both sides of the DE to get

$$(s^2 X(s) - 3s - 1) + 6(sX(s) - 3) + 5X(s) = 0.$$

Solving for  $X(s)$  gives

$$X(s) = \frac{3s+19}{s^2+6s+5} = \frac{3s+19}{(s+1)(s+5)}.$$

Using partial fractions, we obtain

$$X(s) = \frac{4}{s+1} - \frac{1}{s+5}.$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{x(t) = 4e^{-t} - e^{-5t}}$$

(b) We have  $F_e(t) = 30 \sin 2t$  and the initial conditions are

$$x(0) = 0 \quad \text{and} \quad x'(0) = 0.$$

Take the Laplace transform on both sides of the DE to get

$$s^2 X(s) + 6sX(s) + 5X(s) = 30 \left( \frac{2}{s^2 + 4} \right).$$

Solving for  $X(s)$  gives

$$X(s) = \frac{60}{(s^2 + 6s + 5)(s^2 + 4)} = \frac{60}{(s + 1)(s + 5)(s^2 + 4)}.$$

Using partial fractions, we obtain

$$X(s) = 3 \left( \frac{1}{s + 1} \right) - \frac{15}{29} \left( \frac{1}{s + 5} \right) - \frac{72}{29} \left( \frac{s}{s^2 + 4} \right) + \frac{12}{29 \cdot 2} \left( \frac{2}{s^2 + 4} \right).$$

Taking the inverse Laplace transform gives the answer.

$$x(t) = 3e^{-t} - \frac{15}{29} e^{-5t} - \frac{72}{29} \cos 2t + \frac{6}{29} \sin 2t$$

14. The differential equation of the  $LR$  circuit is

$$2i'(t) + 4i(t) = 5e^{-t}$$

with initial condition  $i(0) = 0$ . Take the Laplace transform on both sides of the DE to get

$$2sI(s) + 4I(s) = \frac{5}{s + 1}.$$

Solving for  $I(s)$  gives

$$I(s) = \frac{5}{(2s + 4)(s + 1)}.$$

Using partial fractions, we obtain

$$I(s) = \frac{5}{2(s + 1)} - \frac{5}{2(s + 2)}.$$

Taking the inverse Laplace transform gives the answer.

$$i(t) = \frac{5}{2} e^{-t} - \frac{5}{2} e^{-2t}$$

15. The differential equation of the *LRC* circuit is

$$q''(t) + 2q'(t) + 4q(t) = 50 \cos t$$

with initial condition  $q'(0) = 0$  and  $q(0) = 0$ . Take the Laplace transform on both sides of the DE to get

$$s^2 Q(s) + 2sQ(s) + 4Q(s) = \frac{50s}{s^2 + 1}.$$

Solving for  $Q(s)$  gives

$$Q(s) = \frac{50s}{(s^2 + 1)(s^2 + 2s + 4)}.$$

Using partial fractions, we obtain

$$Q(s) = \frac{50}{13} \left( \frac{3s + 2}{s^2 + 1} \right) - \frac{50}{13} \left( \frac{3s + 8}{s^2 + 2s + 4} \right).$$

By completing the square, we get

$$s^2 + 2s + 4 = (s + 1)^2 + 3.$$

Therefore

$$\begin{aligned} Q(s) &= \frac{50}{13} \left( \frac{3s + 2}{s^2 + 1} \right) - \frac{50}{13} \left( \frac{3(s + 1) + 5}{(s + 1)^2 + 3} \right) \\ &= \frac{150}{13} \left( \frac{s}{s^2 + 1} \right) + \frac{100}{13} \left( \frac{1}{s^2 + 1} \right) - \frac{150}{13} \left( \frac{s + 1}{(s + 1)^2 + 3} \right) - \frac{250}{13\sqrt{3}} \left( \frac{\sqrt{3}}{(s + 1)^2 + 3} \right). \end{aligned}$$

Taking the inverse Laplace transform gives the answer.

$$q(t) = \frac{150}{13} \cos t + \frac{100}{13} \sin t - \frac{150}{13} e^{-t} \cos \sqrt{3}t - \frac{250}{13\sqrt{3}} e^{-t} \sin \sqrt{3}t$$

## 2.4 Step Functions and Impulses

1. We will use the following time shift property.

$$\mathcal{L}\{f(t)\mathcal{U}(t - a)\} = e^{-as}\mathcal{L}\{f(t + a)\}$$

(a) We have  $f(t) = 1 + 2\mathcal{U}(t - 1) - 5\mathcal{U}(t - 2)$ . Then,

$$\mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{2e^{-s}}{s} - \frac{5e^{-2s}}{s}.$$

(b) We have  $f(t) = \sin t - \sin t \mathcal{U}(t - \pi)$ . Then,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{\sin t\} - \mathcal{L}\{\sin t \mathcal{U}(t - \pi)\} \\ &= \frac{1}{s^2 + 1} - e^{-\pi s} \mathcal{L}\{\sin(t + \pi)\} \\ &= \frac{1}{s^2 + 1} - e^{-\pi s} \mathcal{L}\{-\sin t\} \\ &= \boxed{\frac{1}{s^2 + 1} + \frac{e^{-\pi s}}{s^2 + 1}}. \end{aligned}$$

(c) We have  $f(t) = t - t\mathcal{U}(t - 1) + \mathcal{U}(t - 1) - \mathcal{U}(t - 2)$ . Then,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{t\} - \mathcal{L}\{t\mathcal{U}(t - 1)\} + \mathcal{L}\{\mathcal{U}(t - 1)\} - \mathcal{L}\{\mathcal{U}(t - 2)\} \\ &= \frac{1}{s^2} - e^{-s} \mathcal{L}\{t + 1\} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \\ &= \frac{1}{s^2} - e^{-s} \left( \frac{1}{s^2} + \frac{1}{s} \right) + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \\ &= \boxed{\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}}. \end{aligned}$$

(d) We have  $f(t) = t^2 \mathcal{U}(t - 3)$ . Then,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{t^2 \mathcal{U}(t - 3)\} \\ &= e^{-3s} \mathcal{L}\{(t + 3)^2\} \\ &= e^{-3s} \mathcal{L}\{t^2 + 6t + 9\} \\ &= \boxed{e^{-s} \left( \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \right)}. \end{aligned}$$

2. We will use the following time shift property.

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)\mathcal{U}(t - a)$$

(a) If  $F(s) = \frac{1}{s + 3}$ , then  $f(t) = e^{-3t}$ . Therefore,

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s + 3}\right\} = e^{-3(t-2)}\mathcal{U}(t - 2).$$

(b) If  $F(s) = \frac{1}{s^2 + 4} = \frac{1}{2} \left( \frac{2}{s^2 + 4} \right)$ , then  $f(t) = \frac{1}{2} \sin 2t$ . Therefore,

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2 + 4}\right\} &= \frac{1}{2} \sin 2(t - 2\pi)\mathcal{U}(t - 2\pi) \\ &= \boxed{\frac{1}{2} \sin 2t \mathcal{U}(t - 2\pi)}. \end{aligned}$$

(c) Let  $F(s) = \frac{1}{(s-1)(s+2)}$ . Using partial fractions, we get

$$F(s) = \frac{1}{3} \left( \frac{1}{s-1} - \frac{1}{s+2} \right).$$

Since  $f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{3}(e^t - e^{-2t})$ , then

$$\mathcal{L}^{-1} \left\{ \frac{e^{-s}}{(s-1)(s+2)} \right\} = \frac{1}{3} (e^{(t-1)} - e^{-2(t-1)}) \mathcal{U}(t-1).$$

(d) Let  $F(s) = \frac{1}{s(s^2+1)}$ . Using partial fractions, we get

$$F(s) = \frac{1}{s} - \frac{s}{s^2+1}.$$

Since  $f(t) = \mathcal{L}^{-1}\{F(s)\} = 1 - \cos t$ , then

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{e^{-\pi s}}{s(s^2+1)} \right\} &= (1 - \cos(t - \pi)) \mathcal{U}(t - \pi) \\ &= \boxed{(1 + \cos t) \mathcal{U}(t - \pi)}. \end{aligned}$$

**3.** The IVP is

$$y'(t) + 2y(t) = 1 - \mathcal{U}(t-1), \quad y(0) = 3.$$

Take the Laplace transform on both sides of the DE to get

$$(sY(s) - 3) + 2Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}.$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{3}{s+2} + \frac{1}{s(s+2)} (1 - e^{-s}).$$

Let  $G(s) = \frac{1}{s(s+2)}$ . Using partial fractions we obtain

$$G(s) = \frac{1}{2} \left( \frac{1}{s} - \frac{1}{s+2} \right) \implies g(t) = \frac{1}{2} (1 - e^{-2t}).$$

Since

$$Y(s) = \frac{3}{s+2} + G(s) - e^{-s}G(s),$$

we conclude that

$$\begin{aligned} y(t) &= 3e^{-2t} + g(t) - g(t-1)\mathcal{U}(t-1) \\ &= 3e^{-2t} + \frac{1}{2}(1 - e^{-2t}) - \frac{1}{2}(1 - e^{-2(t-1)})\mathcal{U}(t-1) \\ &= \boxed{\frac{1}{2} + \frac{5}{2}e^{-2t} - \frac{1}{2}(1 - e^{-2(t-1)})\mathcal{U}(t-1)}. \end{aligned}$$

We can also express  $y(t)$  as a piecewise defined function as follows.

$$y(t) = \begin{cases} \frac{1}{2} + \frac{5}{2}e^{-2t}, & 0 \leq t < 1 \\ \frac{5}{2}e^{-2t} + \frac{1}{2}e^{-2(t-1)}, & t \geq 1 \end{cases}$$

4. The IVP is

$$y''(t) + y(t) = \sin t \mathcal{U}(t - \pi), \quad y(0) = 1, \quad y'(0) = 0.$$

Take the Laplace transform on both sides of the DE to get

$$\begin{aligned} (s^2Y(s) - s) + Y(s) &= \mathcal{L}\{\sin t \mathcal{U}(t - \pi)\} \\ &= e^{-\pi s} \mathcal{L}\{\sin(t + \pi)\} \\ &= e^{-\pi s} \mathcal{L}\{-\sin t\} \\ &= -\frac{e^{-\pi s}}{s^2 + 1}. \end{aligned}$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{s}{s^2 + 1} - \frac{e^{-\pi s}}{(s^2 + 1)^2}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + 1)^2}\right\} = \frac{\sin t - t \cos t}{2},$$

we get

$$\begin{aligned} y(t) &= \cos t - \left(\frac{\sin(t - \pi) - (t - \pi) \cos(t - \pi)}{2}\right)\mathcal{U}(t - \pi) \\ &= \boxed{\cos t + \left(\frac{\sin t - (t - \pi) \cos t}{2}\right)\mathcal{U}(t - \pi)}. \end{aligned}$$

We can also express  $y(t)$  as a piecewise defined function as follows.

$$y(t) = \begin{cases} \cos t, & 0 \leq t < \pi \\ (1 - \frac{1}{2}(t - \pi)) \cos t + \frac{1}{2} \sin t, & t \geq \pi \end{cases}$$



5. The IVP is

$$y''(t) + y'(t) - 2y(t) = 1 - 2\mathcal{U}(t - 3), \quad y(0) = 0, \quad y'(0) = 0.$$

Take the Laplace transform on both sides of the DE to get

$$s^2Y(s) + sY(s) - 2Y(s) = \frac{1}{s} - \frac{2e^{-3s}}{s}.$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{1 - 2e^{-3s}}{s(s^2 + s - 2)} = \frac{1 - 2e^{-3s}}{s(s-1)(s+2)}.$$

Let  $G(s) = \frac{1}{s(s-1)(s+2)}$ . Using partial fractions we obtain

$$G(s) = \frac{1}{3(s-1)} + \frac{1}{6(s+2)} - \frac{1}{2s} \implies g(t) = \frac{1}{3}e^t + \frac{1}{6}e^{-2t} - \frac{1}{2}.$$

Since

$$Y(s) = G(s) - 2e^{-3s}G(s),$$

we conclude that

$$y(t) = \frac{1}{3}e^t + \frac{1}{6}e^{-2t} - \frac{1}{2} - \left( \frac{2}{3}e^{(t-3)} + \frac{1}{3}e^{-2(t-3)} - 1 \right) \mathcal{U}(t-3).$$

We can also express  $y(t)$  as a piecewise defined function as follows.

$$y(t) = \begin{cases} \frac{1}{3}e^t + \frac{1}{6}e^{-2t} - \frac{1}{2}, & 0 \leq t < 3 \\ \frac{1}{3}(e^t - 2e^{(t-3)}) + \frac{1}{6}(e^{-2t} - 2e^{-2(t-3)}) + \frac{1}{2}, & t \geq 3 \end{cases}$$

6. The IVP is

$$y''(t) + 4y(t) = 1 + \mathcal{U}(t - \pi) - 2\mathcal{U}(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 2.$$

Take the Laplace transform on both sides of the DE to get

$$(s^2Y(s) - 2) + 4Y(s) = \frac{1}{s} + \frac{e^{-\pi s}}{s} - \frac{2e^{-2\pi s}}{s}.$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{2}{s^2 + 4} + \frac{1 + e^{-\pi s} - 2e^{-2\pi s}}{s(s^2 + 4)}.$$

Let  $G(s) = \frac{1}{s(s^2 + 4)}$ . Using partial fractions we obtain

$$G(s) = \frac{1}{4} \left( \frac{1}{s} - \frac{s}{s^2 + 4} \right) \implies g(t) = \frac{1}{4} (1 - \cos 2t).$$

Since

$$Y(s) = \frac{2}{s^2 + 4} + G(s) + e^{-\pi s}G(s) - 2e^{-2\pi s}G(s)$$

we conclude that

$$y(t) = \sin 2t + \frac{1}{4}(1 - \cos 2t) + \frac{1}{4}(1 - \cos 2(t - \pi))\mathcal{U}(t - \pi) - \frac{1}{2}(1 - \cos 2(t - 2\pi))\mathcal{U}(t - 2\pi).$$

Simplifying gives the answer.

$$y(t) = \sin 2t + \frac{1}{4}(1 - \cos 2t) + \frac{1}{4}(1 - \cos 2t)\mathcal{U}(t - \pi) - \frac{1}{2}(1 - \cos 2t)\mathcal{U}(t - 2\pi)$$

We can also express  $y(t)$  as a piecewise defined function as follows.

$$y(t) = \begin{cases} \sin 2t + \frac{1}{4}(1 - \cos 2t), & 0 \leq t < \pi \\ \sin 2t + \frac{1}{2}(1 - \cos 2t), & \pi \leq t < 2\pi \\ \sin 2t, & t \geq 2\pi \end{cases}$$

7. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y'(t)\} + 3\mathcal{L}\{y(t)\} = \mathcal{L}\{\delta(t - 1)\}$$

Since  $y(0) = 2$ , we get

$$(sY(s) - 2) + 3Y(s) = e^{-s}.$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{2}{s + 3} + \frac{e^{-s}}{s + 3}.$$

Taking the inverse Laplace transform gives the answer.

$$y(t) = 2e^{-3t} + e^{-3(t-1)}\mathcal{U}(t - 1)$$

8. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y'(t)\} + 13\mathcal{L}\{y(t)\} = \mathcal{L}\{\delta(t - \pi)\}$$

Since  $y(0) = 2$  and  $y'(0) = 1$ , we get

$$(s^2Y(s) - 2s - 1) + 4(sY(s) - 2) + 13Y(s) = e^{-\pi s}.$$

Solving for  $Y(s)$  gives

$$Y(s) = \frac{2s + 9}{s^2 + 4s + 13} + \frac{e^{-\pi s}}{s^2 + 4s + 13}.$$

By completing the square, we get

$$s^2 + 4s + 13 = (s + 2)^2 + 9.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{2s + 9}{(s + 2)^2 + 9} + \frac{e^{-\pi s}}{(s + 2)^2 + 9} \\ &= \frac{2(s + 2) + 5}{(s + 2)^2 + 9} + \frac{e^{-\pi s}}{(s + 2)^2 + 9} \\ &= 2 \left( \frac{s + 2}{(s + 2)^2 + 9} \right) + \frac{5}{3} \left( \frac{3}{(s + 2)^2 + 9} \right) + \frac{e^{-\pi s}}{3} \left( \frac{3}{(s + 2)^2 + 9} \right). \end{aligned}$$

Taking the inverse Laplace transform gives

$$y(t) = 2e^{-2t} \cos 3t + \frac{5}{3}e^{-2t} \sin 3t + \frac{1}{3}e^{-2(t-\pi)} \sin 3(t - \pi) \mathcal{U}(t - \pi).$$

Since  $\sin 3(t - \pi) = -\sin 3t$ , we obtain the answer

$$\boxed{y(t) = 2e^{-2t} \cos 3t + \frac{5}{3}e^{-2t} \sin 3t - \frac{1}{3}e^{-2(t-\pi)} \sin 3t \mathcal{U}(t - \pi).}$$

9. Take the Laplace transform on both sides of the differential equation.

$$\mathcal{L}\{y''(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{\delta(t - \frac{\pi}{2})\} + 2\mathcal{L}\{\delta(t - \pi)\}$$

Since  $y(0) = 3$  and  $y'(0) = -1$ , we get

$$(s^2 Y(s) - 3s + 1) + Y(s) = e^{-\frac{\pi}{2}s} + 2e^{-\pi s}.$$

Solving for  $Y(s)$  gives

$$\begin{aligned} Y(s) &= \frac{3s - 1}{s^2 + 1} + \frac{e^{-\frac{\pi}{2}s} + 2e^{-\pi s}}{s^2 + 1} \\ &= 3 \left( \frac{s}{s^2 + 1} \right) - \frac{1}{s^2 + 1} + e^{-\frac{\pi}{2}s} \left( \frac{1}{s^2 + 1} \right) + 2e^{-\pi s} \left( \frac{1}{s^2 + 1} \right). \end{aligned}$$

Taking the inverse Laplace transform gives

$$y(t) = 3 \cos t - \sin t + \sin(t - \frac{\pi}{2}) \mathcal{U}(t - \frac{\pi}{2}) + 2 \sin(t - \pi) \mathcal{U}(t - \pi).$$

Since

$$\sin(t - \frac{\pi}{2}) = -\cos t \quad \text{and} \quad \sin(t - \pi) = -\sin t,$$

we obtain the answer

$$\boxed{y(t) = 3 \cos t - \sin t - \cos t \mathcal{U}(t - \frac{\pi}{2}) - 2 \sin t \mathcal{U}(t - \pi).}$$

10. We will use the formula

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt.$$

Since

$$f(t) = \begin{cases} 1, & 0 \leq t < a \\ 0, & a \leq t < 2a \end{cases}$$

and  $T = 2a$ , we have

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t) dt \\ &= \frac{1}{1 - e^{-2as}} \left( \int_0^a e^{-st} f(t) dt + \int_a^{2a} e^{-st} f(t) dt \right) \\ &= \frac{1}{1 - e^{-2as}} \left( \int_0^a e^{-st}(1) dt + \int_a^{2a} e^{-st}(0) dt \right) \\ &= \frac{1}{1 - e^{-2as}} \int_0^a e^{-st} dt \\ &= \left( \frac{1}{1 - e^{-2as}} \right) \left( \frac{1 - e^{-as}}{s} \right) \\ &= \left( \frac{1}{(1 + e^{-as})(1 - e^{-as})} \right) \left( \frac{1 - e^{-as}}{s} \right) \\ &= \boxed{\frac{1}{s(1 + e^{-as})}}. \end{aligned}$$

11. Since  $V(t) = 2 - 2\mathcal{U}(t - 1)$ , the differential equation of the  $LR$  circuit is

$$i'(t) + 4i(t) = 2 - 2\mathcal{U}(t - 1)$$

with initial conditions  $i(0) = 0$ . Take the Laplace transform on both sides of the DE to get

$$sI(s) + 4I(s) = \frac{2}{s} - \frac{2e^{-s}}{s}.$$

Solving for  $I(s)$  gives

$$I(s) = \frac{2}{s(s+4)} - \frac{2e^{-s}}{s(s+4)}.$$

Let  $G(s) = \frac{2}{s(s+4)}$ . Using partial fractions, we obtain

$$G(s) = \frac{1}{2} \left( \frac{1}{s} - \frac{1}{s+4} \right) \implies g(t) = \frac{1}{2} (1 - e^{-4t}).$$

Since

$$I(s) = G(s) - e^{-s}G(s),$$

we can take the inverse Laplace transform to get the current.

(a)

$$i(t) = \frac{1}{2} (1 - e^{-4t}) - \frac{1}{2} (1 - e^{-4(t-1)}) \mathcal{U}(t-1)$$

(b) We can express  $i(t)$  as follows.

$$i(t) = \begin{cases} \frac{1}{2} (1 - e^{-4t}), & 0 \leq t < 1 \\ \frac{1}{2} (e^{-4(t-1)} - e^{-4t}), & t \geq 1 \end{cases}$$

For  $t = 1.5$ , we get

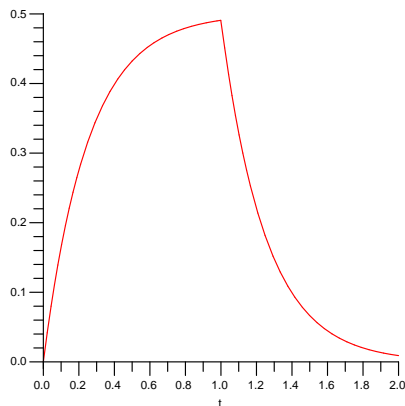
$$i(1.5) = \frac{1}{2} (e^{-4(1.5-1)} - e^{-4(1.5)}) \approx 0.0664.$$

At  $t = 1.5$  seconds, the current is about 0.0664 A.(c) For  $t \geq 1$ , we have

$$i(t) = \frac{1}{2} (e^{-4(t-1)} - e^{-4t}).$$

Therefore,

$$\lim_{t \rightarrow \infty} i(t) = 0.$$

(d) The graph of  $i(t)$  is the following.

Observe that the current  $i(t)$  is continuous even though the voltage is discontinuous at  $t = 1$ .

12. We have

$$\delta_\varepsilon(t-a) = \frac{1}{\varepsilon} \mathcal{U}(t-a) - \frac{1}{\varepsilon} \mathcal{U}(t-(a+\varepsilon)).$$

(a) Taking Laplace transform, we get

$$\begin{aligned}\mathcal{L}\{\delta_\varepsilon(t-a)\} &= \frac{1}{\varepsilon} \left( \frac{e^{-as}}{s} \right) - \frac{1}{\varepsilon} \left( \frac{e^{-(a+\varepsilon)s}}{s} \right) \\ &= \frac{e^{-as}}{s} \left( \frac{1 - e^{-\varepsilon s}}{\varepsilon} \right).\end{aligned}$$

(b) We will use l'Hôpital's rule to compute the limit.

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0} \mathcal{L}\{\delta_\varepsilon(t-a)\} &= \lim_{\varepsilon \rightarrow 0} \frac{e^{-as}}{s} \left( \frac{1 - e^{-\varepsilon s}}{\varepsilon} \right) \\ &= \frac{e^{-as}}{s} \left( \lim_{\varepsilon \rightarrow 0} \frac{1 - e^{-\varepsilon s}}{\varepsilon} \right) \\ &= \frac{e^{-as}}{s} \left( \lim_{\varepsilon \rightarrow 0} \frac{s e^{-\varepsilon s}}{1} \right) \\ &= \frac{e^{-as}}{s} \cancel{(\cancel{s})} \\ &= e^{-as}\end{aligned}$$

## 2.5 Convolution

1. We will use the convolution theorem.

$$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$$

$$(a) \quad \mathcal{L}\{1 * t^4\} = \mathcal{L}\{1\} \mathcal{L}\{t^4\} = \frac{1}{s} \left( \frac{4!}{s^5} \right) = \boxed{\frac{24}{s^6}}$$

$$(b) \quad \text{Observe that } \int_0^t \cos \theta \, d\theta = 1 * \cos t.$$

$$\begin{aligned}\mathcal{L}\left\{ \int_0^t \cos \theta \, d\theta \right\} &= \mathcal{L}\{1 * \cos t\} \\ &= \mathcal{L}\{1\} \mathcal{L}\{\cos t\} \\ &= \frac{1}{s} \left( \frac{\cancel{s}}{s^2 + 1} \right) \\ &= \boxed{\frac{1}{s^2 + 1}}\end{aligned}$$

$$(c) \quad \mathcal{L}\{e^t * \sin t\} = \mathcal{L}\{e^t\} \mathcal{L}\{\sin t\} = \left( \frac{1}{s-1} \right) \left( \frac{1}{s^2+1} \right) = \boxed{\frac{1}{(s-1)(s^2+1)}}$$

(d) Observe that  $\int_0^t \cos \theta \sin(t - \theta) d\theta = \cos t * \sin t$ .

$$\begin{aligned} \mathcal{L}\left\{\int_0^t \cos \theta \sin(t - \theta) d\theta\right\} &= \mathcal{L}\{\cos t * \sin t\} \\ &= \left(\frac{s}{s^2 + 1}\right) \left(\frac{1}{s^2 + 1}\right) \\ &= \boxed{\frac{s}{(s^2 + 1)^2}} \end{aligned}$$

2. We will use the convolution theorem.

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) = \int_0^t f(\theta)g(t - \theta) d\theta$$

(a) Let  $F(s) = \frac{1}{s}$  and  $G(s) = \frac{1}{s - 1}$ , then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 1 \quad \text{and} \quad g(t) = \mathcal{L}^{-1}\{G(s)\} = e^t.$$

Therefore,

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s(s - 1)}\right\} &= \mathcal{L}^{-1}\{F(s)G(s)\} \\ &= f(t) * g(t) \\ &= 1 * e^t \\ &= \int_0^t e^\theta d\theta \\ &= e^\theta \Big|_0^t \\ &= \boxed{e^t - 1}. \end{aligned}$$

(b) Let  $F(s) = \frac{1}{s + 1}$  and  $G(s) = \frac{1}{s - 3}$ , then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-t} \quad \text{and} \quad g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{3t}.$$

Therefore,

$$\begin{aligned}
 \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)(s-3)} \right\} &= \mathcal{L}^{-1} \{F(s)G(s)\} \\
 &= f(t) * g(t) \\
 &= e^{-t} * e^{3t} \\
 &= \int_0^t e^{-(t-\theta)} e^{3\theta} d\theta \\
 &= e^{-t} \int_0^t e^{4\theta} d\theta \\
 &= e^{-t} \left( \frac{e^{4\theta}}{4} \right) \Big|_0^t \\
 &= e^{-t} \left( \frac{e^{4t} - 1}{4} \right) \\
 &= \boxed{\frac{e^{3t} - e^{-t}}{4}}.
 \end{aligned}$$

(c) Let  $F(s) = \frac{1}{s}$  and  $G(s) = \frac{1}{s^2 + 1}$ , then

$$f(t) = \mathcal{L}^{-1} \{F(s)\} = 1 \quad \text{and} \quad g(t) = \mathcal{L}^{-1} \{G(s)\} = \sin t.$$

Therefore,

$$\begin{aligned}
 \mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 + 1)} \right\} &= \mathcal{L}^{-1} \{F(s)G(s)\} \\
 &= f(t) * g(t) \\
 &= 1 * \sin t \\
 &= \int_0^t \sin \theta d\theta \\
 &= -\cos \theta \Big|_0^t \\
 &= \boxed{1 - \cos t}.
 \end{aligned}$$

(d) Let  $F(s) = \frac{1}{s}$  and  $G(s) = \frac{1}{s(s^2 + 1)}$ , then

$$f(t) = \mathcal{L}^{-1} \{F(s)\} = 1 \quad \text{and} \quad g(t) = \mathcal{L}^{-1} \{G(s)\} = 1 - \cos t.$$



Therefore,

$$\begin{aligned}
 \mathcal{L}^{-1} \left\{ \frac{1}{s^2(s^2+1)} \right\} &= \mathcal{L}^{-1} \{F(s)G(s)\} \\
 &= f(t) * g(t) \\
 &= 1 * (1 - \cos t) \\
 &= \int_0^t 1 - \cos \theta \, d\theta \\
 &= (\theta - \sin \theta) \Big|_0^t \\
 &= \boxed{t - \sin t}.
 \end{aligned}$$

3. Since  $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$ , then

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2+1)^2} \right\} = \sin t * \sin t = \int_0^t \sin \theta \sin(t-\theta) \, d\theta$$

By using the identity

$$\sin \alpha \sin \beta = \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

with  $\alpha = \theta$  and  $\beta = t - \theta$ , we get

$$\begin{aligned}
 \sin \theta \sin(t-\theta) &= \frac{1}{2} (\cos(\theta - (t-\theta)) - \cos(\theta + (t-\theta))) \\
 &= \frac{1}{2} (\cos(2\theta - t) - \cos t).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \mathcal{L}^{-1} \left\{ \frac{1}{(s^2+1)^2} \right\} &= \frac{1}{2} \int_0^t (\cos(2\theta - t) - \cos t) \, d\theta \\
 &= \frac{1}{2} \left( \frac{\sin(2\theta - t)}{2} - \theta \cos t \right) \Big|_0^t \\
 &= \frac{1}{2} \left( \frac{\sin t}{2} - t \cos t \right) - \frac{1}{2} \left( \frac{\sin(-t)}{2} - 0 \right) \\
 &= \frac{1}{2} (\sin t - t \cos t)
 \end{aligned}$$

We can now apply the scaling property

$$\mathcal{L}\{f(\omega t)\} = \frac{1}{\omega} F\left(\frac{s}{\omega}\right)$$

to the function  $f(t) = \frac{1}{2}(\sin t - t \cos t)$  to get

$$\begin{aligned} \mathcal{L}\left\{\frac{1}{2}(\sin \omega t - \omega t \cos \omega t)\right\} &= \frac{1}{\omega} \left( \frac{1}{\left(\left(\frac{s}{\omega}\right)^2 + 1\right)^2} \right) \\ &= \frac{\omega^3}{(s^2 + \omega^2)^2}. \end{aligned}$$

We can conclude that

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + \omega^2)^2}\right\} = \frac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}.}$$

4. First observe that

$$\int_0^t e^{-\theta} f(t - \theta) d\theta = e^{-t} * f(t).$$

The equation is then equivalent to

$$f(t) = t + e^{2t} + e^{-t} * f(t).$$

Take Laplace transform on both sides to get

$$F(s) = \frac{1}{s^2} + \frac{1}{s-2} + \frac{F(s)}{s+1}.$$

Solve for  $F(s)$  to get

$$F(s) = \frac{1}{s^2} + \frac{1}{s^3} + \frac{s+1}{s(s-2)}$$

Using partial fractions, we obtain

$$\frac{s+1}{s(s-2)} = \frac{-1/2}{s} + \frac{3/2}{s-2}.$$

Therefore,

$$F(s) = \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{2} \left( \frac{1}{s} \right) + \frac{3}{2} \left( \frac{1}{s-2} \right)$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{f(t) = t + \frac{t^2}{2} - \frac{1}{2} + \frac{3}{2}e^{2t}}$$

5. The equation is of the form

$$f(t) * f(t) = t^3.$$

Take Laplace transform to get

$$F(s) \cdot F(s) = F(s)^2 = \frac{6}{s^4} \implies F(s) = \pm \frac{\sqrt{6}}{s^2}.$$

By taking the inverse Laplace transform we obtain the two answers

$$\boxed{f(t) = \pm \sqrt{6}t.}$$

6. First observe that

$$\int_0^t e^{-2\theta} y(t - \theta) d\theta = e^{-2t} * y(t).$$

The equation is then equivalent to

$$y'(t) + e^{-2t} * y(t) = 1.$$

Since  $y(0) = 0$ , we can take Laplace transform on both sides to get

$$sY(s) + \frac{Y(s)}{s+2} = \frac{1}{s}.$$

Solve for  $Y(s)$  to get

$$Y(s) = \frac{s+2}{s(s+1)^2}.$$

Using partial fractions, we deduce

$$Y(s) = \frac{2}{s} - \frac{2}{s+1} - \frac{1}{(s+1)^2}.$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = 2 - 2e^{-t} - te^{-t}}$$



# Appendix **A**

## Formulas and Properties

### A.1 Table of Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}$	$e^{at}$	$\frac{1}{s-a}$
$t^n$	$\frac{n!}{s^{n+1}}$	$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
$\frac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}$	$\frac{1}{(s^2 + \omega^2)^2}$	$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t)$	1	$\delta(t-a)$	$e^{-as}$

## A.2 Properties of Laplace Transforms

### 1. Definition

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

### 2. Linearity

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$$

### 3. Time differentiation

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \quad \mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$$

### 4. Frequency differentiation

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

### 5. Time shift

$$\mathcal{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as} \mathcal{L}\{f(t+a)\}$$

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\mathcal{U}(t-a)$$

### 6. Frequency shift

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

### 7. Convolution

$$f(t) * g(t) = \int_0^t f(\theta)g(t-\theta) d\theta \implies \mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$$

### 8. Integration

$$\mathcal{L}\left\{\int_0^t f(\theta) d\theta\right\} = \frac{F(s)}{s}$$

### 9. Periodic function

$$f(t) \text{ has period } T \implies \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$$

### 10. Scaling

$$\mathcal{L}\{f(\omega t)\} = \frac{1}{\omega} F\left(\frac{s}{\omega}\right)$$

## A.3 Trigonometric Identities

### 1. Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1 \quad 1 + \tan^2 \theta = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta$$

### 2. Symmetry Identities

$$\sin(-\theta) = -\sin \theta \quad \cos(-\theta) = \cos \theta \quad \tan(-\theta) = -\tan \theta$$

### 3. Sum and Difference Identities

$$\begin{aligned} \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta & \sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta & \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ \tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} & \tan(\alpha - \beta) &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \end{aligned}$$

### 4. Double-Angle Identities

$$\begin{aligned} \cos 2\theta &= \cos^2 \theta - \sin^2 \theta & \sin 2\theta &= 2 \sin \theta \cos \theta \\ &= 1 - 2 \sin^2 \theta & \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \\ &= 2 \cos^2 \theta - 1 \end{aligned}$$

### 5. Power-Reducing Identities

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

### 6. Product-To-Sum Identities

$$\begin{aligned} \sin \alpha \cos \beta &= \frac{1}{2} (\sin(\alpha - \beta) + \sin(\alpha + \beta)) \\ \sin \alpha \sin \beta &= \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) \\ \cos \alpha \cos \beta &= \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta)) \end{aligned}$$





# Appendix **B**

## Partial Fractions

### B.1 Partial Fractions

Consider a function  $F(s)$  expressed as a quotient of two polynomials

$$F(s) = \frac{P(s)}{Q(s)}$$

where  $Q(s)$  has a degree smaller than the degree of  $P(s)$ . The method of partial fractions can be summarized as follows.

1. Completely factor  $Q(s)$  into factors of the form

$$(ps + q)^m \quad \text{and} \quad (as^2 + bs + c)^n$$

where  $as^2 + bs + c$  is an irreducible quadratic.

2. For *each* factor of the form  $(ps + q)^m$ , the partial fractions decomposition must include the following  $m$  terms.

$$\frac{A_1}{ps + q} + \frac{A_2}{(ps + q)^2} + \cdots + \frac{A_m}{(ps + q)^m}.$$

3. For *each* factor of the form  $(as^2 + bs + c)^n$ , the partial fractions decomposition must include the following  $n$  terms.

$$\frac{B_1s + C_1}{as^2 + bs + c} + \frac{B_2s + C_2}{(as^2 + bs + c)^2} + \cdots + \frac{B_ns + C_n}{(as^2 + bs + c)^n}.$$

4. Find the values of all the constants.

## B.2 Cover-up Method

There are several different ways to determine the constants in a partial fractions decomposition. When the denominator has distinct roots, we can use the cover-up method. Let's illustrate it with some examples.

**Example 1.** Consider

$$\frac{s^2 + 5s + 4}{(s-1)(s-3)(s+2)} = \frac{A}{s-1} + \frac{B}{s-3} + \frac{C}{s+2} \quad (\text{B.1})$$

To find  $A$ , we could use the following two steps.

1. Multiply both sides of equation (B.1) by  $(s-1)(s-3)(s+2)$  to obtain

$$s^2 + 5s + 4 = A(s-3)(s+2) + B(s-1)(s+1) + C(s-1)(s-3).$$

2. Set  $s = 1$  and solve for  $A$ .

$$1 + 5 + 4 = A(-2)(3) + 0 + 0 \implies A = -\frac{5}{3}$$

The cover-up method combines these two steps in a single one as follows. To find  $A$ , we cover up  $(s-1)$  and set  $s = 1$  in the left-hand side of (B.1).

$$A = \frac{s^2 + 5s + 4}{\cancel{(s-1)}(s-3)(s+2)} \Big|_{s=1} = -\frac{5}{3}$$

We can find constants  $B$  and  $C$  in the same way. To find  $B$ , we cover up  $(s-3)$  and set  $s = 3$ .

$$B = \frac{s^2 + 5s + 4}{(s-1)\cancel{(s-3)}(s+2)} \Big|_{s=3} = \frac{14}{5}$$

To find  $C$ , we cover up  $(s+2)$  and set  $s = -2$ .

$$C = \frac{s^2 + 5s + 4}{(s-1)(s-3)\cancel{(s+2)}} \Big|_{s=-2} = -\frac{2}{15}$$

If the denominator includes irreducible quadratic terms, the cover-up method works if we use complex numbers as in the following examples.

**Example 2.** Consider

$$\frac{s}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}.$$

Since  $s+1=0$  if  $s=-1$ , then

$$A = \frac{s}{\cancel{(s+1)}(s^2+4)} \Big|_{s=-1} = -\frac{1}{5}$$

Since  $s^2+4=0$  if  $s=2j$ , then

$$\begin{aligned} (Bs+C)|_{s=2j} &= \frac{s}{(s+1)\cancel{(s^2+4)}} \Big|_{s=2j} \\ 2Bj+C &= \frac{2j}{1+2j} \\ &= \frac{2j}{(1+2j)} \left( \frac{1-2j}{1-2j} \right) \\ &= \frac{2j+4}{5}. \end{aligned}$$

The real and imaginary parts of the complex numbers on both sides are equal.

We conclude that

$$2B = \frac{2}{5} \implies B = \frac{1}{5}$$

and

$$C = \frac{4}{5}.$$

**Example 3.** Consider

$$F(s) = \frac{s^2}{(s+2)(s^2+6s+13)}.$$

By completing the square, we get

$$s^2 + 6s + 13 = (s+3)^2 + 4.$$

We can look for a partial fractions decomposition of  $F(s)$  in the form

$$\frac{s^2}{(s+2)(s^2+6s+13)} = \frac{A}{s+2} + \frac{B(s+3)+C}{(s+3)^2+4}.$$

Observe that by using  $B(s+3)+C$  instead  $Bs+C$ , we make it a little bit easier since we do not have to solve a linear system to find  $B$  and  $C$ .

Since  $s+2=0$  if  $s=-2$ , then

$$A = \frac{s^2}{\cancel{(s+2)}(s^2+6s+13)} \Big|_{s=-2} = \frac{4}{5}$$

Since  $s^2+6s+13=(s+3)^2+4=0$  if  $s=-3+2j$ , then

$$\begin{aligned} (B(s+3)+C)|_{s=-3+2j} &= \frac{s^2}{(s+2)\cancel{(s^2+6s+13)}} \Big|_{s=-3+2j} \\ 2Bj+C &= \frac{(-3+2j)^2}{(-3+2j)+2} \\ &= \frac{5-12j}{-1+2j} \\ &= \frac{5-12j}{-1+2j} \left( \frac{-1-2j}{-1-2j} \right) \\ 2Bj+C &= \frac{2j-29}{5}. \end{aligned}$$

The real and imaginary parts of the complex numbers on both sides are equal. We conclude that

$$2B = \frac{2}{5} \implies B = \frac{1}{5}$$

and

$$C = -\frac{29}{5}.$$

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