Constant Coefficients and Cauchy-Euler Equations

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Constant Coefficients.
A homogeneous second-order constant coefficient equation is of the form

$$ay''(x) + by'(x) + cy(x) = 0. \tag{1}$$

To solve the equation, plug $\boxed{y = e^{mx}}$ into (1) and deduce the characteristic equation

$$am^2 + bm + c = 0. \tag{2}$$

The general solution depends on the nature of the roots of (2).

Roots	**General Solution**
m_1 and m_2 are distinct real roots	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
m is the unique real root	$y = c_1 e^{mx} + c_2 x e^{mx}$
$m_{1,2} = \alpha \pm \beta i$ are complex conjugates roots	$y = e^{\alpha x}\big(c_1 \sin(\beta x) + c_2 \cos(\beta x)\big)$

Cauchy-Euler.
A homogeneous second-order Cauchy-Euler equation is of the form

$$ax^2 y''(x) + bxy'(x) + cy(x) = 0. \tag{3}$$

To solve the equation, plug $\boxed{y = x^m}$ into (3) and deduce the characteristic equation

$$am^2 + (b-a)m + c = 0. \tag{4}$$

The general solution depends on the nature of the roots of (4).

Roots	**General Solution**
m_1 and m_2 are distinct real roots	$y = c_1 x^{m_1} + c_2 x^{m_2}$
m is the unique real root	$y = c_1 x^m + c_2 x^m \ln x$
$m_{1,2} = \alpha \pm \beta i$ are complex conjugates roots	$y = x^{\alpha}\big(c_1 \sin(\beta \ln x) + c_2 \cos(\beta \ln x)\big)$

Note. The solutions given for equation (1) are valid over the whole real line. The solutions given for equation (3) are valid over the interval $(0, \infty)$.

The similarity between the solutions of the constant coefficients and Cauchy-Euler equations can be explained as follows. Start from the Cauchy-Euler equation

$$ax^2y''(x) + bxy'(x) + cy(x) = 0. \tag{5}$$

Introduce a new variable t defined by

$$t = \ln x \quad \Longleftrightarrow \quad x = e^t.$$

Observe that the x-interval $(0, \infty)$ transforms into the t-interval $(-\infty, \infty)$. Assume that $y(x)$ is a solution of (5) and let

$$y(x) = y(e^t) = Y(t).$$

Using the chain-rule we deduce that

$$\frac{dy}{dx} = \frac{dY}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dY}{dt} \implies x\frac{dy}{dx} = \frac{dY}{dt}.$$

Similarly, we deduce that

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[\frac{dY}{dt}\frac{dt}{dx}\right]\\ &= \frac{d}{dx}\left[\frac{dY}{dt}\right]\frac{dt}{dx} + \frac{dY}{dt}\frac{d^2t}{dx^2}\\ &= \left(\frac{d^2Y}{dt^2}\frac{dt}{dx}\right)\frac{dt}{dx} + \frac{dY}{dt}\frac{d^2t}{dx^2}\\ &= \frac{1}{x^2}\frac{d^2Y}{dt^2} - \frac{1}{x^2}\frac{dY}{dt}.\end{aligned}$$

Therefore,

$$x^2\frac{d^2y}{dx^2} = \frac{d^2Y}{dt^2} - \frac{dY}{dt}.$$

By substitution in (5) we deduce

$$a\big(Y''(t) - Y'(t)\big) + bY'(t) + cY(t) = 0$$

which simplifies to

$$aY''(t) + (b-a)Y'(t) + cY(t) = 0. \tag{6}$$

We conclude that equation (5) is equivalent to the constant coefficients equation (6) in the sense that

$$Y(t) \text{ is a solution of (6)} \iff y(x) = Y(\ln x) \text{ is a solution of (5)}.$$