

## Method of Undetermined Coefficients

**Problem.** Find a particular solution  $y_p$  of the constant coefficients linear equation

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = g(x).$$

We assume that

$$g(x) = [\text{polynomial}] \times [\text{exponential}] \times [\text{sinusoid}].$$

More precisely, we assume that

$$g(x) = p(x) e^{\alpha x} (a \cos(\omega x) + b \sin(\omega x)),$$

where  $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m$ .

**Basic Guess.**

- If  $\omega = 0$ , so that  $g(x) = p(x) e^{\alpha x}$ , we look for  $y_p$  in the form

$$y_p = (A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m) e^{\alpha x}$$

for the undetermined coefficients  $A_0, A_1, A_2, \dots, A_m$ .

- If  $\omega \neq 0$ , we look for  $y_p$  in the form

$$y_p = (A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m) e^{\alpha x} \cos(\omega x) + (B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m) e^{\alpha x} \sin(\omega x)$$

for the undetermined coefficients  $A_0, A_1, A_2, \dots, A_m$  and  $B_0, B_1, B_2, \dots, B_m$ .

We always choose  $m$  in both cases above to match the degree of the polynomial  $p(x)$ . If  $g(x)$  does not include an exponential term, simply set  $\alpha = 0$ .

**Examples.**

$g(x)$	Form of $y_p$
5	$A$
$3x^2$	$Ax^2 + Bx + C$
$5x^3 + 2x - 3$	$Ax^3 + Bx^2 + Cx + D$
$3e^{4x}$	$Ae^{4x}$
$(5x + 3)e^{2x}$	$(Ax + B)e^{2x}$
$x^2 e^{6x}$	$(Ax^2 + Bx + C)e^{6x}$
$5 \cos(2x)$	$A \cos(2x) + B \sin(2x)$
$e^{-2x}(\cos(3x) + 2 \sin(3x))$	$Ae^{-2x} \cos(3x) + Be^{-2x} \sin(3x)$
$6x^2 \sin(5x)$	$(Ax^2 + Bx + C) \cos(5x) + (Dx^2 + Ex + F) \sin(5x)$
$x e^{3x} \cos(2x)$	$(Ax + B)e^{3x} \cos(2x) + (Cx + D)e^{3x} \sin(2x)$

If  $g(x) = g_1(x) + g_2(x) + \cdots + g_k(x)$ , we apply the procedure separately to each  $g_i(x)$  to produce a particular solution

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}.$$

**Example.** If

$$g(x) = x^2 + 4xe^{3x} + 5 \cos(2x),$$

then the basic form of  $y_p$  is

$$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p_1}} + \underbrace{(Dx + E)e^{3x}}_{y_{p_2}} + \underbrace{F \cos(2x) + G \sin(2x)}_{y_{p_3}}.$$

**“Bad Case”.** If any of the  $y_{p_i}$  contains terms that duplicate terms in the homogeneous solution  $y_c$ , then that  $y_{p_i}$  must be multiplied by  $x^n$ , where  $n$  is the smallest positive integer that eliminates the duplication.

**Example 1.** Consider the equation

$$y'' - 6y' + 8y = 3e^{2x}.$$

Since the roots of  $m^2 - 6m + 8 = 0$  are  $m = 2$  and  $m = 4$ , then  $y_c = c_1e^{2x} + c_2e^{4x}$ . Since  $g(x) = 3e^{2x}$ , the basic form of  $y_p$  is  $y_p = Ae^{2x}$ . Since  $Ae^{2x}$  duplicates the term  $c_1e^{2x}$  in  $y_c$ , we have a “bad case”. We should multiply  $Ae^{2x}$  by  $x$  and use the following

$$\boxed{y_p = Axe^{2x}}$$

**Example 2.** Consider the equation

$$y'' - 6y' + 9y = 5xe^{3x}.$$

The only root of  $m^2 - 6m + 9 = 0$  is  $m = 3$ . Then,  $y_c = c_1e^{3x} + c_2xe^{3x}$ . Since  $g(x) = 5xe^{3x}$ , the basic form of  $y_p$  is  $y_p = (Ax + B)e^{3x}$ . Since this  $y_p$  duplicates terms in  $y_c$ , we have a “bad case”. Multiplying by  $x$  would still involve terms duplicated in  $y_c$ . We should then multiply by  $x^2$  and use the following.

$$\boxed{y_p = (Ax^3 + Bx^2)e^{3x}}$$

**Example 3.** Consider the equation

$$y'' + 9y = 5e^x + 2 \sin(3x).$$

The roots of  $m^2 + 9 = 0$  are  $m = \pm 3i$ . Then,  $y_c = c_1 \cos(3x) + c_2 \sin(3x)$ . Since  $g(x) = g_1(x) + g_2(x)$ , where  $g_1(x) = 5e^x$  and  $g_2(x) = 2 \sin(3x)$ , then  $y_p = y_{p_1} + y_{p_2}$ . The basic forms of  $y_{p_1}$  and  $y_{p_2}$  are

$$y_{p_1} = Ae^x \quad \text{and} \quad y_{p_2} = B \cos(3x) + C \sin(3x).$$

Since  $y_{p_2}$  duplicates  $y_c$ , we have a “bad case”. We should multiply  $y_{p_2}$  by  $x$  and use the following .

$$\boxed{y_p = Ae^x + Bx \cos(3x) + Cx \sin(3x)}$$

Observe that we do not multiply  $y_{p_1}$  by  $x$ .

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