Section 3.1 : Sequences and Series

Thursday, February 08, 2024 1:29 PM

sequence $\equiv$ an ordered list of numbers (often with a pattern)

examples :

pattern?

① $2, 5, 8, \ldots$ add 3

② $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{256}$ mult by $\frac{1}{2}$

③ $\underset{①}{1}, \underset{②}{4}, \underset{③}{9}, \underset{④}{16}, \underset{⑤}{25}, \ldots, 100$ n^2

④ $5, -5, -15, \ldots$ add -10

infinite sequence : ① ④

finite sequence : ② ③

notation :

$a_1, a_2, a_3, \ldots, a_n, \ldots$

↑ if you are starting your count from one

$a_0, a_1, a_2, \ldots a_n, \ldots$

↑ if you are counting from zero

$a_m, a_{m+1}, a_{m+2}, \ldots, a_n, \ldots$

↑ what's the previous term?

a_{n-1}

three ways to define a sequence

① list all of the terms, <u>or</u>

list enough of the terms to set up the pattern (minimum number is three, thoug for complicated patterns, may need far more)

note: If sequence is finite, then you need to give either the final term <u>or</u> the total number of terms

② give a general formula for a_n

③ give a recursive formula for a_n

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general formula: formula that gives a_n by way of n only

example: $\boxed{a_n = \underbrace{3n-1}_{} \quad \text{for } n \geq 1}$

right hand side only has n as a variable - no other variables

what are the first three terms of this sequence?

answer:

$a_1 = 3(1) - 1 = 2$

$a_2 = 3(2) - 1 = 5$

$a_3 = 3(3) - 1 = 8$

by the way, what's a_{100}? $a_{100} = 3(100) - 1 = 299$

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example: write all terms of the sequence

$$a_n = 2^n + 1 \quad \text{for} \quad 0 \le n \le 4$$

answer:
$a_0 = 2^0 + 1 = 2$
$a_1 = 2^1 + 1 = 3$
$a_2 = 2^2 + 1 = 5$
$a_3 = 2^3 + 1 = 9$
$a_4 = 2^4 + 1 = 17$

note: how many terms are there in ~~total~~? 5

if k is the total number of terms, then

$$k = \underset{\substack{\uparrow \\ \text{final} \\ \text{index}}}{n} - \underset{\substack{\uparrow \\ \text{first} \\ \text{index}}}{m} + 1$$

example: what is the n^{th} term of the sequence

$$\underset{①}{1},\ \underset{②}{\sqrt{2}},\ \underset{③}{\sqrt{3}},\ \underset{④}{2},\ \underset{⑤}{\sqrt{5}},\ \underset{⑥}{\sqrt{6}},\ \ldots$$

answer: $\boxed{a_n = \sqrt{n} \quad \text{for} \quad n \ge 1}$

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the recursive formula: gives the next term by way of the previous term

first or initial term

example:
$$\begin{cases} \boxed{a_1} = 2 \\ \boxed{a_n} = \boxed{a_{n-1}} + 3 \quad \text{for } n \geq 2 \end{cases}$$

next term

previous term

give the first three terms

answer:
$a_1 = 2$
$a_2 = a_1 + 3 = 2 + 3 = 5$
$a_3 = a_2 + 3 = 8$

$\underline{2}, \underline{5}, \underline{8}$

example: Fibonacci sequence

what is the recursive formula for this sequence?

1, 1, 2, 3, 5, 8, 13, ...

answer:
$$\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 3 \end{cases}$$

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digression: will not be tested

what is the general formula for this sequence?

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{for } n \geq 1$$

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series:

examples: $2 + 5 + 8 + \dots$

$5 + 15 + 25 + \dots + 105$ ← the sum of the terms of a finite sequence is just a number

so a series is just the sum of the terms of a sequence

notation:

letter ess ↓

S_k = sum of the first k terms of a sequence

(if the series is finite, could be the sum of all terms)

S_∞ = sum of all terms of an infinite series

note: if k is large, calculating S_k could be annoying! but we will learn more efficient methods later

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consider the following series

$16 + 20 + 24 + \dots$

for this series, calculate S_3 and S_5

answer: $S_3 = 16 + 20 + 24 = 60$

$$S_5 = 16 + 20 + 24 + 28 + 32 = 120$$

note: it does not matter what the starting index is when calculating S_k

$$S_3 = a_1 + a_2 + a_3$$ if starting index is 1

$$= a_0 + a_1 + a_2$$ " 0

$$= a_{17} + a_{18} + a_{19}$$ 17

$$\boxed{k = n - m + 1}$$

sigma notation:

$$\sum_{n=1}^{4} (3n+1) = \overset{\textcircled{1}}{(3\cdot1+1)} + \overset{\textcircled{2}}{(3\cdot2+1)} + \overset{\textcircled{3}}{(3\cdot3+1)} + \overset{\textcircled{4}}{(3\cdot4+1)}$$

$$= 4 + 7 + 10 + 13$$

Greek letter sigma

$$= 34$$

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$$\sum_{i=0}^{2} 3^i = \overset{\textcircled{0}}{3^0} + \overset{\textcircled{1}}{3^1} + \overset{\textcircled{2}}{3^2}$$

$$= 1 + 3 + 9 \boxed{= 13}$$

example : evaluate

$$\sum_{j=2}^{8} 7 = \overset{\textcircled{2}}{7} + \overset{\textcircled{3}}{7} + \overset{\textcircled{4}}{7} + \overset{\textcircled{5}}{7} + \overset{\textcircled{6}}{7} + \overset{\textcircled{7}}{7} + \overset{\textcircled{8}}{7}$$

$$= 49$$

note: how many terms?

$$k = n - m + 1$$
$$= 8 - 2 + 1 = 7$$

example: write the following using sigma notation:

$$\frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

answer: $$\sum_{i=6}^{\infty} \frac{1}{i}$$

$\underline{\underline{\text{or}}}$ $$\sum_{k=1}^{\infty} \frac{1}{k+5}$$

$\underline{\underline{\text{or}}}$ $$\sum_{m=0}^{\infty} \frac{1}{m+6}$$

digression: will not be tested

why do we care?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$3! = 3 \cdot 2 \cdot 1$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$