Section 3.3: Geometric Sequences and Series

Wednesday, February 14, 2024 12:04 PM

examples:

pattern?

① 7, 14, 28, 56, ..., 114688 — multiply by 2

② 100, 20, 4, 4/5, ... — mult by 1/5

③ 1/2, 1/4, 1/8, 1/16, ... 1/256 — mult by 1/2

④ 24, -16, 32/3, -64/9, ... — mult by $-2/3$

how do you find this?

take any term except the first and divide by previous term

$$\frac{-16}{24} = \frac{-2\cdot\cancel{8}}{3\cdot\cancel{8}} = \frac{-2}{3}$$

$$\frac{32}{3} \div (-16) = \frac{\overset{2}{\cancel{32}}}{3}\cdot\frac{1}{-\cancel{16}} = \frac{-2}{3}$$

geometric sequence $\equiv$ a sequence in which the term is equal to the previous term multiplied by a constant

common ratio

r

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recursive formula:

example: give a recursive formula for the sequence

$$100, 20, 4, 4/5, \ldots$$

answer: geometric with $r = \frac{1}{5}$

$$\begin{cases} a_0 = 100 \\ a_n = \frac{1}{5} a_{n-1} & \text{for } n \geq 1 \end{cases}$$

or $n > 0$

or $n = 1, 2, 3, \ldots$

r

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general formula:

example: find a general formula for $7, 14, 28, \ldots$

1	2	3			n
$7,$	$14,$	$28,$	$\ldots$	,	a_n
$7,$	$7 \cdot 2,$	$7 \cdot 2^2,$		,	$7 \cdot 2^{n-1}$

the general formula for this particular sequence is

$$\boxed{a_n = 7 \cdot 2^{n-1} \quad \text{for } n \geq 1}$$

for geometric,

$$\boxed{a_n = a_m r^{n-m} \quad \text{for} \quad n \geq m}$$

example: consider the sequence 5, 15, 45, ...

find a) the twelfth term
b) the fifthieth term

answer: geometric with $r = 3$

a) twelfth term: if $m = 1$, we want a_{12}
(if $m = 0$, want a_{11})

$$a_n = a_m r^{n-m}$$

$$a_{12} = 5 \cdot 3^{12-1} = 5 \cdot 3^{11} = 885\,735$$

b) fiftieth term

$$a_{50} = 5 \cdot 3^{49} = 1.196 \times 10^{24}$$

example: give the general formula for

80, −160, 320, ...

answer: geometric with $r = -2$

$$a_n = a_m r^{n-m}$$

$$\boxed{a_n = 80(-2)^{n-1} \quad \text{for } n \geq 1}$$

note: $a_n = 80 - 2^{n-1}$ is not correct

$= 80 \cdot -2^{n-1}$ is not correct

why not? $-2^2 = -(2^2) = -4$

$(-2)^2 = 4$

in WeBWorK, write 80(-2)^(n-1)

geometric series: $7 + 14 + 28 + \ldots$

first term

$$S_k = \frac{a_m(1 - r^n)}{1 - r} \quad \text{where } k \geq 1 \text{ and } r \neq 1$$

sum of the first k terms

common ratio

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example: for the series $2 + 10 + 50 + \ldots$

find the sum of the first
a) twelve terms
b) fifty terms

a) geometric with $r = 5$

$$S_k = \frac{a_m(1 - r^k)}{1 - r}$$

$$= \frac{2(1 - 5^{12})}{1 - 5}$$

$$= 1.22 \times 10^8 \quad \text{or} \quad 122\ 070\ 312$$

b) want S_{50}

$$S_{50} = \frac{2(1 - 5^{50})}{1 - 5}$$

$$= 4.44 \times 10^{34}$$

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what about $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$?

how do we make this work?

$$S_k = \frac{a_m(1 - r^k)}{1 - r}$$

$$S_\infty = a_m \frac{(1 - r^\infty)}{1 - r}$$

but what is r^∞?

here we have $r = \frac{1}{4}$

let's consider $\left(\frac{1}{4}\right)^n$

as $n \to \infty$, $\left(\frac{1}{4}\right)^n \to 0$

as $n \to \infty$, $\left(\frac{1}{4}\right) \to 0$

↑ approaches

but <u>only</u> for

$-1 < r < 1$

or $|r| < 1$

then $S_\infty = \dfrac{a_m(1 - \cancel{r^\infty}^{\,0})}{1-r}$

$$\boxed{S_\infty = \frac{a_m}{1-r}}$$ for $-1 < r < 1$

so, back to our original question,

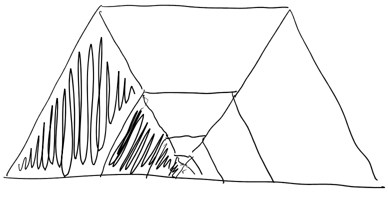
$1/4 + 1/16 + 1/64 + \ldots$

geometric with $r = \frac{1}{4}$

is $-1 < r < 1$? ✓

$$S_\infty = \frac{a_m}{1-r}$$

$$= \frac{\frac{1}{4}}{1 - 1/4} = \frac{1/4}{3/4} = \frac{1}{4} \cdot \frac{4}{3} = \frac{1}{3}$$



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example: evaluate $24 - 16 + 32/3 - \ldots$

answer:

geometric
with $r = -\frac{2}{3} = -0.\overline{6}$

$r = \frac{-16}{24} = \frac{-2 \cdot \cancel{8}}{3 \cdot \cancel{8}} = \frac{-2}{3}$

$= \frac{32/3}{-16} = \frac{32}{3} \cdot \frac{1}{-16} = \frac{-2}{3}$

is $-1 < r < 1$? ✓

$S_\infty = \frac{a_m}{1-r} = \frac{24}{1-(-2/3)} = \frac{24}{5/3} = 24 \times \frac{3}{5}$

$= \frac{72}{5}$ or 14.4

example: evaluate $12 + 18 + 27 + \dots$

$\cancel{S_\infty = \frac{a_m}{1-r} = \frac{12}{1-3/2} = \frac{12}{-1/2} = 12 \cdot \frac{1}{-2} = -24}$

is $-1 < r < 1$? NO

S_∞ is undefined

S_∞ does not exist (DNE)

note: for series like
$12 + 18 + 27 + \dots$, you could also say $S_\infty \to \infty$

$-12 - 18 - 27 - \dots$, $S_\infty \to -\infty$

but for alternating series

$12 - 18 + 27 - \ldots$

<u>must</u> say ONE

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example: evaluate

$$\sum_{j=0}^{\infty} 75\left(\frac{3}{5}\right)^j = \underset{0}{75} + \underset{1}{75(3/5)} + \underset{2}{75(3/5)^2} + \ldots$$

$$= 75 + 45 + 27 + \ldots$$

geometric with $r = \frac{3}{5}$ or 0.6

is $-1 < r < 1$? ✓

$$S_\infty = \frac{a_m}{1-r} = \frac{75}{1 - 3/5} = \frac{75}{2/5} = 187.5$$

summary for geometric:

$$\boxed{\begin{aligned} a_n &= a_m r^{n-m} \\ S_k &= \frac{a_m(1-r^k)}{1-r} \\ S_\infty &= \frac{a_m}{1-r} \end{aligned}}$$

$\Leftarrow$ for $n \geq m$

$\Leftarrow$ where $k = n-m+1$

$\leftarrow$ where $-1 < r < 1$