# Section 3.1: Sequences and Series

Friday, October 06, 2023 11:23 AM

sequence ≡ an ordered list of numbers (often with a pattern)

examples:

pattern?

① $2, 5, 8, \ldots$ adding 3

② $1/2, 1/4, 1/8, 1/16, \ldots 1/256$ mult by $1/2$

③ ①1, ②4, ③9, ④16, ⑤25, …. 100 $n^2$

④ $5, -5, -15, \ldots$ add $-10$

infinite sequences: ① ④
finite ② ③

notation:

$a_1, a_2, a_3, \ldots, a_n, \ldots$

↑ if you start counting at one

$a_0, a_1, a_2, a_3, \ldots a_n, \ldots$

↑ if you start counting at zero

$a_m, a_{m+1}, a_{m+2}, \ldots a_n, \ldots$ ← what is the previous term?

$a_{n-1}$

three ways to define a sequence:

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① list all of the terms, <u>or</u> list enough terms to set up the pattern (minimum number is three, though will need more for complicated patterns)

note: if the sequence is finite, then you need to give either the final term or the total number of terms.

② give a general formula for $a_n$

③ give a recursive formula for $a_n$

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general formula: formula that gives $a_n$ by way of $n$ only

example: $\boxed{a_n = \underbrace{3n-1}_{} \quad \text{for } n \geq 1}$

right hand side only has $n$ as a variable, no other variables

What are the first three terms of this sequence?

answer:
$a_1 = 3(1) - 1 = 2$
$a_2 = 3(2) - 1 = 5$
$a_3 = 3(3) - 1 = 8$

whats $a_{100}$? $a_{100} = 3(100) - 1 = 299$

example: write all terms of the sequence

$$a_n = 2^n + 1 \quad \text{for } 0 \leq n \leq 4$$

answer: $a_0 = 2^0 + 1 = 2$
$a_1 = 2^1 + 1 = 3$
$a_2 = 2^2 + 1 = 5$
$a_3 = 2^3 + 1 = 9$
$a_4 = 2^4 + 1 = 17$

note: how many terms in total are there?

if $k$ is the total number of terms, then

$$k = \underset{\substack{\uparrow \\ \text{final} \\ \text{index}}}{n} - \underset{\substack{\uparrow \\ \text{starting} \\ \text{index}}}{m} + 1$$

example: what is $a_n$ for the sequence

① ② ③ ④ ⑤ ⑥
$1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \ldots$

$$\boxed{a_n = \sqrt{n} \quad \text{for } n \geq 1}$$

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## Section 3.1: cont'd

the recursive formula: gives the next term by way of the previous term (or terms) example: $\begin{cases} a_1 = 2 \\ a_n = a_{n-1} + 3 \quad \text{for } n \geq 2 \end{cases}$ ← first term

the next term → $a_n$; $a_{n-1}$ ← previous

give the first three terms

answer: $a_1 = 2$

$a_2 = a_1 + 3 = 2 + 3 = 5$

$a_3 = a_2 + 3 = 5 + 3 = 8$

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example: Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, ...

what is the recursive form of this sequence?

answer: $\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 3 \end{cases}$

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digression: (will not be tested) what is the general formula for this sequence?

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{for } n \geq 1$$

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series:

examples: $2 + 5 + 8 + \ldots$

$5 + 15 + 25 + \ldots + 105$ ← the sum of

$5 + 15 + 25 + \dots + 105$ ← the sum of all terms in a finite series is just a number

So a series is just the sum of the terms of a sequence

notation: $S_k$ = sum of the first $k$ terms of a series

(if series is finite, could be the sum of <u>all</u> terms)

(also known as the $k^{th}$ partial sum)

$S_\infty$ = sum of all terms of an infinite series

note: when calculating $S_k$, if $k$ is large, then this calculation could be annoying.

we will learn more efficient methods later

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consider the following series:

$16 + 20 + 24 + \dots$

for this series, calculate $S_3$ and $S_5$

answer: $S_3 = 16 + 20 + 24 = 60$

$S_5 = 16 + 20 + 24 + 28 + 32 = 120$

note: it does not matter what the starting index is when calculating $S_k$

$S_3 = a_1 + a_2 + a_3$ if starting index is 1

$= a_0 + a_1 + a_2$

$= a_{17} + a_{18} + a_{19}$

remember: $$\underset{\text{total number of terms}}{k} = \overset{\text{final index}}{n} - \underset{\text{starting index}}{m} + 1$$

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sigma notation

$$\sum_{n=1}^{4} (3n+1) = \overset{\textcircled{1}}{(3 \cdot 1 + 1)} + \overset{\textcircled{2}}{(3 \cdot 2 + 1)} + \overset{\textcircled{3}}{(3 \cdot 3 \cdot 1)} + \overset{\textcircled{4}}{(3 \cdot 4 + 1)}$$

$$= 4 + 7 + 10 + 13$$

$$= 34$$

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## Section 3.1: cont'd 2023/10/12

$$\sum_{i=0}^{2} 3^i = \overset{\textcircled{0}}{3^0} + \overset{\textcircled{1}}{3^1} + \overset{\textcircled{2}}{3^2}$$

$$= 1 + 3 + 9$$

$$= 13$$

evaluate

$$\sum_{j=2}^{8} 7 = \overset{\textcircled{2}}{7} + \overset{\textcircled{3}}{7} + \overset{\textcircled{4}}{7} + \overset{\textcircled{5}}{7} + \overset{\textcircled{6}}{7} + \overset{\textcircled{7}}{7} + \overset{\textcircled{8}}{7}$$

$$= 49$$

note: how many terms? rule is

$$\underset{\substack{\uparrow \\ \text{total number of terms}}}{k} = \underset{\substack{\uparrow \\ \text{final index}}}{n} - \underset{\substack{\nwarrow \\ \text{starting index}}}{m} + 1$$

example: write the following series in sigma notation

$$\overset{\textcircled{1}}{\frac{1}{6}} + \overset{\textcircled{2}}{\frac{1}{7}} + \overset{\textcircled{3}}{\frac{1}{8}} + \cdots$$

answer: $$\sum_{i=6}^{\infty} \frac{1}{i}$$

or $$\sum_{k=1}^{\infty} \frac{1}{k+5}$$

$$= \sum^{\infty} \frac{1}{\phantom{x}}$$

or $$\sum_{j=0}^{\infty} \frac{1}{j+6}$$

digression: will not be tested

why do we care?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$3! = 3 \cdot 2 \cdot 1$$
$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$