Section 3.1: Sequences and Series

Sequences

Let's start out with the definition of a sequence:

sequence: an ordered list of numbers, often with a definite pattern

Recall that in a set, order doesn't matter so this is one way that a sequence differs from a set. Also, repetition doesn't matter in a set but does in a sequence: if a number is repeated in a sequence, it isn't considered a "duplicate" and cannot be removed without changing the sequence.

Sequences, like sets, can be finite or infinite. If a sequence is finite, then either the last term or the number of terms must be specified so that it's clear where the sequence stops.

Example

Which of the following sequences are infinite? Which are finite?

a) 7, 11, 15, 19, …

b) 1, 4, 9, 16, 25, 36, … 100

c) $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \frac{1}{256}$

Answer

b) and c) are finite, because their last terms are given. a), however, goes on forever so is infinite.

To begin with, let's examine some sequences in detail. We will begin by looking for patterns in each sequence.

Example

What is the pattern for the following sequences? What is the next term for each sequence?

a) 7, 11, 15, 19, …

b) 1, 4, 9, 16, 25, 36, … 100

c) $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \frac{1}{256}$

d) 3, –6, 12, –24, …

e) 3, –6, –15, –24, …

Answer

a) The pattern is that you add 4 to the previous term to get the next term. The next term is then 23.

b) The pattern is that if you say that “1” is the first term and “4” is the second term, then n^2 will be the nth term. So the next term after 36 is 49.

c) The pattern is to divide each term by two (or multiply by a half) to get the next term. So the term after 1/16 will be 1/32.

d) The pattern is to multiply each term by –2 to get the next term. The next term is then 48.

e) The pattern is to subtract 9 from the previous term, so the next one is –33.

Note that in this previous example, the last two sequences looked very similar for three of their first four terms. However, the third term is different so the pattern for the two sequences is not the same and subsequent terms could look very different.

Notation

We will use the notation a_n for the nth term in a sequence, where n is the index. For example, the first term would then be a_1, the second term a_2, and so on. The index n, then, is a positive integer (or a natural number, if you like).

Other notations may start their counting with a_o being the first term. For the purposes of this course, we’ll stick to starting at $n = 1$.

Defining a Sequence

There are three ways to define a sequence:

1) List all of the terms, or enough terms to set up the pattern. If the sequence is finite, then either the last term or the number of terms must be given.

2) Give a general formula for the nth term.

3) Give a recursive formula for the nth term.

Let’s look at examples of each type. For instance, the sequences 7, 11, 15, 19, … and 1, 4, 9, 16, 25, 36, … 100 are examples of sequences defined by listing the terms.

General Formula

A general formula is a formula that gives a_n as a function of n only. Let's look at the following examples to examine some sequences defined in this way.

Example

Give the first four terms of the sequence given by the general formula $a_n = 4n + 3$.

Answer
$a_n = 4n + 3$, so

$a_1 = 4 \times 1 + 3 = 7$

$a_2 = 4 \times 2 + 3 = 11$

$a_3 = 4 \times 3 + 3 = 15$

$a_4 = 4 \times 4 + 3 = 19$

The first four terms are then 7, 11, 15, and 19. This is the same sequence that was given as part a) in the first two examples of this section.

Example

Give all terms of the sequence given by the formula $a_n = \left(\frac{1}{3}\right)^n$ for $1 \le n \le 5$.

Answer

This is a finite sequence, since restrictions have been placed on the values of n. The terms are then:

$$a_1 = \left(\frac{1}{3}\right)^1 = \frac{1}{3}$$

$$a_2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$a_3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

$$a_4 = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$$

$$a_5 = \left(\frac{1}{3}\right)^5 = \frac{1}{243}$$

You can see from the previous examples that the general formula allows you to calculate a_n for any value of n. The very useful thing about the general formula is that you don't need to know the previous term to calculate a particular term. For instance, if you want to know the 50^{th} term of the sequence 7, 11, 15, 19, …, you can determine that the pattern is to add 4 to the previous term to get the next term. However, to get the 50^{th} term, you'd have to calculate the 49^{th} first, but the 49^{th} requires the 48^{th}, and so on. But if you instead use the expression $a_n = 4n+3$, which gives the same sequence, then the 50^{th} term is just

$$a_n = 4n+3$$
$$a_{50} = 4\cdot 50+3 = 203$$

and there's no need to calculate preceding terms. Handy!

Recursive Definition

A recursive formula gives a formula for the next term in terms of the previous one. For example, in our old friend 7, 11, 15, 19, … , the next term is found by adding 4 to the previous term: $a_n = a_{n-1}+4$. However, that's not enough information to uniquely define the series because you don't know where to start. A complete definition must include the first term also. Therefore, the recursive definition for our old friend 7, 11, 15, 19, … would be

$$\begin{cases} a_1 = 7 \\ a_n = a_{n-1}+4 \end{cases}$$

Recursive definitions, then, must specify the first term or terms and also the rule which allows you to calculate the next term from the previous term or terms.

Example

Calculate the first four terms of the sequence given by

$$\begin{cases} a_1 = 3 \\ a_n = \left(a_{n-1}-1\right)^2+10 \end{cases}$$

Answer

The first term is already given, $a_1 = 3$. Then

$$a_2 = (3-1)^2+10 = 2^2+10 = 14$$
$$a_3 = (14-1)^2+10 = 13^2+10 = 179$$
$$a_4 = (179-1)^2+10 = 178^2+10 = 31694$$

Example

Give a recursive formula for the sequence 2, 6, 18, 54, …

Answer

The pattern is that the next term equals the previous term times three. Therefore,

$$\begin{cases} a_1 = 2 \\ a_n = 3a_{n-1} \end{cases}$$

Recursive definitions have the same drawback that we've seen before: if we want to know the 200^{th} term, we need to calculate the 199^{th} first, and so on. Only the general formula allows us to calculate each term directly without knowing the previous one.

Fibonacci sequence

The Fibonacci sequence is the most famous example of a recursive sequence:

1, 1, 2, 3, 5, 8, 13, …

The pattern can be quite difficult to spot – you get the next term from the **sum** of the two previous terms. The recursive formula for this sequence is therefore

$$\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \end{cases}$$

Here, the first two terms must be given to start off with so that you are then able to calculate the third term from the previous two.

Series

A series is the sum of a sequence, which can be finite or infinite. Here are two examples:

a) 16 + 20 + 24 + 28 + … 64

b) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+...$

Notation

The sum of the first *n* terms of a sequence is denoted by S_n (also sometimes called the *n*th partial sum). If the series is finite, it could be the sum of **all** of the terms. S_∞ is how we write the sum of an infinite series, like the second example above.

Example

For the series 16 + 20 + 24 + 28 + … 64, calculate S_3 and S_4.

Answer

$S_3 = 16 + 20 + 24 = 60$

$S_4 = 16 + 20 + 24 + 28 = 88$

However, it's easy to see that this method becomes very cumbersome for large values of n. We'll develop some more efficient methods in the next two sections.

Sigma notation

It's easy to take a sequence in list form and transform it into a series by changing all of the commas to + signs. However, what if you are given the general formula instead? For example, let's take 7, 11, 15, 19, … which we know to be $a_n = 4n + 3$. Since the general form is so useful for finding a_n when n is large, it would be nice if we could retain that information while writing our sum.

To do so, we'll introduce a new notation called "sigma notation". It uses the Greek letter sigma (the uppercase one): Σ, which is commonly used to mean "sum of".

Let's look at an example of sigma notation and discuss what all of the parts mean. Consider the following

$$\sum_{i=1}^{5}(4i+3)$$

The letter i is an index here, and it runs from the value given at the bottom of the sigma to the number at the top of the sigma in steps of 1. Here, i runs from 1 to 5. We are summing, then, the value of $4i + 3$ for each value of i as it runs from 1 to 5:

$$\begin{aligned}
& \qquad\quad\; i=1 \qquad\quad i=2 \qquad\quad i=3 \qquad\quad i=4 \qquad\quad i=5 \\
\sum_{i=1}^{5}(4i+3) &= (4\times1+3)+(4\times2+3)+(4\times3+3)+(4\times4+3)+(4\times5+3) \\
&= \quad 7 \quad + \quad 11 \quad + \quad 15 \quad + \quad 19 \quad + \quad 23 \\
&= 75
\end{aligned}$$

Let's look at more examples.

Example

Calculate $\sum_{i=1}^{3}(2i-5)$

Answer

$$\begin{aligned}&\qquad\quad i=1 \qquad\quad i=2 \qquad\quad i=3\\ \sum_{i=1}^{3}(2i-5) &= (2\times1-5)+(2\times2-5)+(2\times3-5)\\ &= \quad -3 \quad + \quad -1 \quad + \quad 1\\ &= -3\end{aligned}$$

Example

Calculate $\sum_{j=6}^{9}(8-j)^2$

Answer

$$\begin{aligned}&\qquad j=6 \qquad j=7 \qquad j=8 \qquad j=9\\ \sum_{j=6}^{9}(8-j)^2 &= (8-6)^2+(8-7)^2+(8-8)^2+(8-9)^2\\ &= \quad 4 \quad + \quad 1 \quad + \quad 0 \quad + \quad 1\\ &= 6\end{aligned}$$

Example

Calculate $\sum_{k=12}^{16}3$

Answer

$$\begin{aligned}&\quad k=12 \quad k=13 \quad k=14 \quad k=15 \quad k=16\\ \sum_{j=12}^{16}3 &= \quad 3 \quad + \quad 3 \quad + \quad 3 \quad + \quad 3 \quad + \quad 3\\ &= 15\end{aligned}$$

The tricky thing about the last one is deciding how many terms there are. You may, as is shown above, write out all of the possible values of the index. Or you may use the following nifty rule:

$$\# \text{ terms} = \text{last} - \text{first} + 1$$

For instance, the last example had the index running from 12 to 16. The number of terms, then, for that series is 16 – 12 + 1 = 5.

Example

Write the following series in sigma notation:

$$4+9+16+25+\ldots 100$$

Answer

Let's pick our index first. If we want to be lazy, instead of starting our index at 1, we could start at 2 and our series would be

$$\sum_{k=2}^{10} k^2$$

Other acceptable answers would involve changing our starting point for the index to give $\sum_{j=1}^{9}(j+1)^2$ or $\sum_{i=0}^{8}(i+2)^2$ or even $\sum_{l=157}^{165}(l-155)^2$ if 157 happens to be your favourite number.

Example

Write the following sequence in sigma notation:

$$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots$$

Answer

$$\sum_{j=3}^{\infty}\frac{1}{j}$$

To write an infinite series in sigma notation, you just replace the final value of the index with ∞.

Section 3.1: Sequences and Series

Exercises

Predict the next three terms of the following sequences.

1. 18, 16, 14, …

2. 1, 4, 9, 16, …

3. 12, 24, 48, 96, …

4. 144, 36, 9, …

5. $1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, ...$

6. 5, –10, 20, …

7. 13, 25, 37, 49, …

8. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$

Predict the general term (or n^th term a_n) of the following sequences.

9. 1, 4, 9, 16, …

10. $1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, ...$

11. 2, 4, 6, 8, …

12. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$

Find the first four terms of the following recursively defined sequences.

13. $\begin{cases} a_1 = 2 \\ a_n = a_{n-1} + 5 \end{cases}$

14. $\begin{cases} a_1 = 10 \\ a_n = 3a_{n-1} \end{cases}$

15. $\begin{cases} a_1 = 2 \\ a_2 = 3 \\ a_n = a_{n-1} \times a_{n-2} \end{cases}$

16. $\begin{cases} a_1 = 2 \\ a_n = \dfrac{1}{a_{n-1}} + 1 \end{cases}$

In each of the following, the general formula for the nth term of a sequence is given. Find the first four terms.

17. $a_n = 3n - 5$

18. $a_n = 3^{n-2}$

19. $a_n = n!$

20. $a_n = \dfrac{1}{n^2}$

In each of the following, the general formula for the nth term of a sequence is given. Calculate the specified terms.

21. $a_n = 5\left(2^{n+1}\right)$; a_7

22. $a_n = 4n + 15$; a_{100}

23. $a_n = \dfrac{n+2}{n+1}$; a_{2500}

24. $a_n = 2n^3$; a_{10}

Calculate S_3 and S_6 for the following series.

25. 3 + 6 + 9 + …

26. 1 + 4 + 9 + 16 + …

27. 5 –10 + 20 – 40 + …

28. 5 + 3 + 1 + …

Write out each sum in full and then evaluate.

29. $\sum_{n=3}^{7} n$

30. $\sum_{j=4}^{10} (-1)^j$

31. $\sum_{i=0}^{4} 2^i$

32. $\sum_{k=20}^{25} (3k - 10)$

Write each series in sigma notation.

33. 1 + 8 + 27 + 64 + … 1000

34. $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$

35. 2 + 4 + 6 + 8 + …

36. 2 + 4 + 6 + 8

Evil alert!

37. (nasty) Write the sequence 1, 4, 9, 16, … using a **recursive** definition.

38. (tricksy) Write the sequence 1, 2, 6, 24, … using a **general** formula.

39. (challenging) What's the next term in the sequence 4, 5, 20, 100, 2000 … ? What's the recursive formula for this sequence?

Section 3.1: Sequences and Series

Solutions

1. 12, 10, 8 (pattern is to subtract 2)
2. 25, 36, 49 (*n*th term is equal to n^2
3. 192, 384, 768 (multiply by 2)
4. $\frac{9}{4}, \frac{9}{16}, \frac{9}{64}$ (divide by 4)
5. $\sqrt{7}, 2\sqrt{2}, 3$ (*n*th term is $\sqrt{n}$)
6. –40, 80, –160 (multiply by –2)
7. 61, 73, 85 (add 12)
8. $\frac{1}{6}, \frac{1}{7}, \frac{1}{8}$
9. $a_n = n^2$
10. $a_n = \sqrt{n}$
11. $a_n = 2n$
12. $a_n = \frac{1}{n+1}$
13. 2, 7, 12, 17
14. 10, 30, 90, 270
15. 2, 3, 6, 18
16. $2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}$
17. –2, 1, 4, 7
18. $\frac{1}{3}, 1, 3, 9$
19. 1, 2, 6, 24

20. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$

21. $a_7 = 1280$

22. $a_{100} = 415$

23. $a_{2500} = \frac{2502}{2501}$

24. $a_{10} = 2000$

25. $S_3 = 18$, $S_6 = 63$

26. $S_3 = 14$, $S_6 = 91$

27. $S_3 = 15$, $S_6 = -105$

28. $S_3 = 9$, $S_6 = 0$

29. $\sum_{n=3}^{7} n = 3 + 4 + 5 + 6 + 7 = 25$

30. $\sum_{j=4}^{10} (-1)^j = 1 + (-1) + 1 + (-1) + 1 + (-1) + 1 = 1$

31. $\sum_{i=0}^{4} 2^i = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31$

32. $\sum_{k=20}^{25} 3k - 10 = 50 + 53 + 56 + 59 + 62 + 65 = 345$

33. $\sum_{i=1}^{10} i^3$

34. $\sum_{j=2}^{\infty} \frac{1}{j}$

35. $\sum_{k=1}^{\infty} 2k$

36. $\sum_{k=1}^{4} 2k$

37. You could either do $\begin{cases} a_1 = 1 \\ a_n = \left(\sqrt{a_{n-1}} + 1\right)^2 \end{cases}$ or another possibility is $\begin{cases} a_1 = 1 \\ a_n = a_{n-1} + 2n - 1 \end{cases}$

38. $a_n = n!$

39. The next term is 200,000. $\begin{cases} a_1 = 4 \\ a_2 = 5 \\ a_n = a_{n-1} \times a_{n-2} \end{cases}$

Section 3.2: Arithmetic Sequences and Series

Arithmetic Sequences

Let's start out with a definition:

> arithmetic sequence: a sequence in which the next term is found by adding a constant (the common difference *d*) to the previous term

Here are some examples of arithmetic sequences:

a) 7, 11, 15, 19, …

b) 11, 4, –3, –10, … –59

c) 12, 12.3, 12.6, 12.9, …

The first one has a common difference of 4, the second –7, and the third 0.3. Note that in each of them, we can find the common difference *d* by taking **any** term and subtracting the previous term from it.

Example

For the following sequences, state whether each of them is arithmetic.

a) –3, –10, –17, –24, …

b) 4, 5, 7, 10, …

c) 2, 4, 8, 16, …

d) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \frac{1}{20}$

Answer

a) Yes, because the common difference *d* is –7.

b) No, because you're not adding the same number each time.

c) No, because you're multiplying by 2 to get the next term, not adding.

d) No, because the difference between each pair of terms is different.

Again, you can define an arithmetic sequence in one of three ways: by listing the terms, by giving a recursive definition, or by giving a general definition.

Recursive Definitions for Arithmetic Sequences

Let's look first at an example.

Example

Give a recursive definition for the sequence 2, 10, 18, 26, …

Answer

Recall that a recursive definition has two parts: listing the first term and giving the pattern. In this case, the pattern is adding $d = 8$ to the previous term to get the next term. The recursive definition is therefore

$$\begin{cases} a_1 = 2 \\ a_n = a_{n-1} + 8 \end{cases}$$

More generally, the recursive formula for **any** arithmetic sequence is

$$\begin{cases} a_1 = \text{< insert value here>} \\ a_n = a_{n-1} + d \end{cases}$$

General Formulae for Arithmetic Sequences

Let's examine the previous example in more detail to see if we can recognize any patterns and come up with a general formula. Rewriting each term, we get

$$\begin{array}{llll} 2, & 10, & 18, & 26, \quad ... \\ 2, & 2+6, & 2+6\times 2, & 2+6\times 3, ... \end{array}$$

So the 3rd term equals the first plus 6 times 2, the 4th term equals the first plus 6 times 3, and the nth term will equal the first plus 6 times $(n - 1)$. More generally, the nth term will equal the first plus d times $(n - 1)$. In other words,

$$a_n = a_1 + (n-1)d$$

for any **arithmetic sequence**.

Example

Write a general formula for the sequence 2, 10, 18, 26, …

Answer

This sequence is arithmetic with the first term 2 and common difference 8.

$$a_n = a_1 + (n-1)d$$
$$a_n = 2 + (n-1)8$$
$$a_n = 2 + 8n - 8$$
$$a_n = 8n - 6$$

The general formula is then that $a_n = 8n - 6$.

Example

What is the 50th term in the sequence in the sequence 2, 10, 18, 26, … ?

Answer

This is the same sequence from the previous example. We may then use the formula we derived, $a_n = 8n - 6$, with $n = 50$.

$$a_n = 8n - 6$$
$$a_n = 8 \times 50 - 6$$
$$a_n = 400 - 6$$
$$a_n = 394$$

The 50th term is 394.

Example

What is the common difference in the arithmetic sequence in which the first term is 18 and the twelfth term is –59?

Answer

$$a_n = a_1 + (n-1)d$$
$$-59 = 18 + (12-1)d$$
$$-77 = 11d$$
$$d = -7$$

The common difference is –7.

Example

Which term has a value of 404 in the sequence –37, –28, –19, …?

Answer

So a_1 is –37 and d is +9. Then we want to find the value of n for which a_n equals 404.

$$
\begin{aligned}
&a_n = a_1 + (n-1)d \\
&404 = -37 + (n-1)9 \\
&441 = 9(n-1) \\
&49 = n-1 \\
&n = 50
\end{aligned}
$$

The **fiftieth** term is 404.

Example

Find the first four terms of the arithmetic sequence in which the thirteenth term is 97 and the fiftieth term is 393.

Answer

So a_{13} = 97 and a_{50} = 393.

Then we find that

$$
\begin{aligned}
&a_n = a_1 + (n-1)d \\
&97 = a_1 + (13-1)d \\
&97 = a_1 + 12d
\end{aligned}
$$

However, this has two unknowns, a_1 and d. Let's look at a_{50}:

$$
\begin{aligned}
&a_n = a_1 + (n-1)d \\
&393 = a_1 + (50-1)d \\
&393 = a_1 + 49d
\end{aligned}
$$

We now have two unknowns, but two equations, giving us the system

$$
\begin{cases}
97 = a_1 + 12d \\
393 = a_1 + 49d
\end{cases}
$$

Solving this system, we first multiply the top equation by negative 1:

$$-97 = -a_1 - 12d$$
$$393 = a_1 + 49d$$

And then add the two equations together, so that the a_1 terms cancel out.

$$296 = 37d$$
$$d = 8$$

Now we substitute into one of the original equations:

$$97 = a_1 + 12d$$
$$97 = a_1 + 12 \times 8$$
$$97 = a_1 + 96$$
$$a_1 = 1$$

Since $a_1 = 1$ and $d = 8$, our sequence is then 1, 9, 17, 25, …

Arithmetic Series

Recall that S_n is the sum of the first n terms of a series. Let's look at a couple of examples of arithmetic series to see if we can identify any patterns.

Suppose we wish to take some partial sums of the series 2 + 10 + 18 + 26 + …. Let's first calculate S_6. We could just find the first six terms and add them up, but notice the following:

$$S_6 = 2 + 10 + 18 + 26 + 34 + 42$$

The sum of the first and last numbers is 44. The sum of the second and second-to-last is also 44. So is the sum of the third and third-last. So when you take the terms in pairs, each pair has the same sum, $\left(a_1 + a_n\right)$, and there are $n/2$ pairs in total. Then

$$S_n = \frac{n}{2}\left(a_1 + a_n\right).$$

What if, however, there are an odd number of terms? Let's also calculate S_7:

$$S_7 = 2 + 10 + 18 + 26 + 34 + 42 + 50$$

The sum of the first and last is 52, as is the sum of the each "inner pair". Notice that the middle, unpaired value, is ½ of 52. So in a sense, the middle term is ½ of a pair, for a total of 3½ pairs. But that's just 7/2, which is our $n/2$ in the original formula! So we're still good. The relationship

$$S_n = \frac{n}{2}(a_1 + a_n)$$

still works, for both odd and even values of n.

Example

Find the sum of the first forty terms of the series 2 + 10 + 18 + 26 + …. .

Answer

This is just the same sequence as before, with $a_1 = 2$ and $d = 8$. In order to use our previous formula, however, we need to calculate a_{40} before we can calculate S_{40}.

$$a_n = a_1 + (n-1)d$$
$$a_{40} = 2 + 39 \times 8 = 314$$

So,

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{40} = \frac{40}{2}(2 + 314) = 20 \times 316 = 6320$$

The sum of the first forty terms is 6320. (Much easier than writing out the first forty terms and adding them up!)

In the previous example, we used the formula for a_n to calculate the last term and put its value into the formula for S_n. We could do that in a more general way:

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) \\ &= \frac{n}{2}\left[a_1 + \left(a_1 + (n-1)d\right)\right] \\ &= \frac{n}{2}\left[2a_1 + (n-1)d\right] \end{aligned}$$

and the last expression, which gives S_n as a function of the first term, the number of terms, and the common difference, can also be used to evaluate series.

Example

Find the sum of the first one hundred terms of the sequence 5, –6, –17, –26, … .

Answer

This sum will just be 5 + –6 + –17 + –26 + …, with a_1 = 5 and d = –11.

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$
$$S_{100} = \frac{100}{2}\left[2\times 5 + 99\times(-11)\right] = -53950$$

Example

Calculate $\sum_{j=3}^{18} 5n+10$.

Answer

The first term will be for j=3 and will equal 5(3)+10=25. Next is j=4 and will equal 5(4)+10=30, j=5 equaling 5(5)=35, and so on. The last term will be for j=18 and will equal 5(18)+10=100.

In other words, our series is 25+30+35+…100. Is it arithmetic? Yes, with common difference d = 5.

What else do we need for our calculation? The number of terms equals (last – first +1), so is (18-3+1)=16. Then

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{16} = \frac{16}{2}(25+100) = 1000$$

Example

Pat the math instructor asks her students to do five word problems the first week, six the second week, seven the third week, and so on, increasing the number of word problems each week by one.

a) How many word problems will diligent students be doing in the last week of classes (the 11^{th} week)?

b) How many word problems will diligent students have completed during the course of the term (11 weeks)?

Answer

a) The number of word problems is a sequence: 5, 6, 7, … . In fact, it's an arithmetic sequence with $a_1 = 5$ and $d = 1$. In the eleventh week, then,

$$a_n = a_1 + (n-1)d$$
$$a_{11} = 5 + 10 \times 1 = 15$$

Diligent students will solve 15 word problems in the last week of classes.

b) The **total** number of word problems solved is

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{11} = \frac{11}{2}(5 + 15) = 110$$

Diligent students will have solved 110 word problems in total.

Summary

For an **arithmetic** sequence, the *n*th term is given by $a_n = a_1 + (n-1)d$

For an **arithmetic** series, the sum of the first n terms (nth partial sum) is $S_n = \frac{n}{2}(a_1 + a_n)$ or $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

Section 3.2: Arithmetic Sequences and Series

Exercises

State whether the following sequences are arithmetic or not. If they are, state the first term and common difference.

1. 8, 9, 11, 13, 16, …

2. –3, –10, –17, –24, …

3. 3, 6, 12, 24, …

4. 1, 2, 6, 24, …

5. 81, 72, 63, 54, …

6. $1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \frac{9}{4}, ...$

Give both the general formula (or n^{th} term a_n) and the recursive formula for the following arithmetic sequences. For the general formula, be sure to simplify your answer.

7. 1, 3, 5, 7, …

8. 5, –6, –17, –28, …

9. –40, –37, –34, –31, …

10. 24, 28, 32, 36, …

For the following arithmetic sequences, calculate a_{50} and a_{261}.

11. 18, 16, 14, 12, …

12. 12, 12.3, 12.6, 12.9, …

State whether the following recursively defined sequences are arithmetic or not. (Is there an easy way to tell?)

13. $\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + 4 \end{cases}$

14. $\begin{cases} a_1 = 12 \\ a_n = 2a_{n-1} \end{cases}$

15. $\begin{cases} a_1 = 75 \\ a_n = a_{n-1} - 20 \end{cases}$

16. $\begin{cases} a_1 = 6 \\ a_n = a_{n-1} + 1 \end{cases}$

17. $\begin{cases} a_1 = 7 \\ a_n = 2 - a_{n-1} \end{cases}$

18. $\begin{cases} a_1 = 3 \\ a_n = \left(a_{n-1}\right)^2 \end{cases}$

19. For the following sequence, calculate the 201st term: 5, 15, 25, 35, …

20. For the following sequence, which term equals 137? 1, 9, 17, 25, …

21. What is the common difference for the arithmetic sequence with a_1 = 200 and $a_{12} = -240$?

22. Calculate the first term for the arithmetic sequence with common difference 7 whose sixteenth term is 102.

23. Calculate the first four terms of the arithmetic sequence in which the sixth term is 17 and the sixtieth term is 179.

24. Calculate the first four terms of the arithmetic sequence in which the one hundredth term is 403 and the sixty-fourth term is 259.

25. Give a general formula for the arithmetic sequence in which the twentieth term is -107 and the thirty-fifth term is –152.

26. Give a recursive formula for the arithmetic sequence in which the eleventh term is 44 and the fifty-second term is 290.

27. Calculate S_{20} for the series 100 + 97 + 94 + …

28. Evaluate the series 12 + 17 + 22 + … 82.

29. Evaluate the series 144 + 138 + 132 + … 78.

30. Calculate S_{100} for the series –20 + –16 + –12 + …

31. Calculate the sum of the odd numbers between 100 and 500.

32. Find the sum of the natural numbers from 50 to 125, inclusive.

Calculate the following sums.

33. $\sum_{k=0}^{53} 5k - 1$

34. $\sum_{j=10}^{92} 6j$

35. $\sum_{i=30}^{140} 2i + 7$

36. $\sum_{k=3}^{502} 17 - 3k$

37. In a supermarket display, there are 37 cans in the bottom layer, 35 in the next layer up, 33 in the next, and so on. How many layers are there if there are 7 cans in the top row?

38. In the previous problem, how many cans are there altogether?

39. In an old-fashioned theatre, there are 25 seats in the first row, 26 in the next, 27 in the one after, and so on. If there are 20 rows in total, how many seats are there altogether?

Section 3.2: Arithmetic Sequences and Series

Solutions

1. not arithmetic

2. yes, $d = -7$

3. no

4. no

5. yes, $d = -9$

6. yes, $d = \frac{1}{4}$

7. $a_n = 2n - 1$ and $\begin{cases} a_1 = 1 \\ a_n = a_{n-1} + 2 \end{cases}$

8. $a_n = 16 - 11n$ and $\begin{cases} a_1 = 5 \\ a_n = a_{n-1} - 11 \end{cases}$

9. $a_n = 3n - 43$ and $\begin{cases} a_1 = -40 \\ a_n = a_{n-1} + 3 \end{cases}$

10. $a_n = 4n + 20$ and $\begin{cases} a_1 = 24 \\ a_n = a_{n-1} + 4 \end{cases}$

11. $a_n = 20 - 2n$, so $a_{50} = -80$ and $a_{261} = -502$

12. $a_n = 11.7 + 0.3n$, so $a_{50} = 26.7$ and $a_{261} = 90$

13. first four terms are 5, 9, 13, 17, so arithmetic with $d = 4$

14. first four terms are 12, 24, 48, 96, so not arithmetic

15. first four terms are 75, 55, 35, 15, so arithmetic with $d = -20$

16. first four terms are 6, 7, 8, 9, so arithmetic with $d = 1$

17. first four terms are 7, –5, 7, –5, so not arithmetic

18. first four terms are 3, 9, 81, 6561, so not arithmetic

19. $a_n = 10n - 5$, so $a_{201} = 2005$

20. $a_n = 8n - 7$, so $n = 18$

21. $d = -40$

22. $a_1 = -3$

23. $a_1 = 2$ and $d = 3$, so the first four terms are 2, 5, 8, 11

24. $a_1 = 7$ and $d = 4$, so the first four terms are 7, 11, 15, 19

25. $a_n = -3n - 47$

26. $\begin{cases} a_1 = -16 \\ a_n = a_{n-1} + 6 \end{cases}$

27. $S_{20} = 1430$

28. $S_{15} = 705$

29. $S_{12} = 1332$

30. $S_{100} = 17800$

31. $S_{200} = 60000$

32. $S_{76} = 6650$

33. $S_{53} = 7101$

34. $S_{83} = 25398$

35. $S_{111} = 19647$

36. $S_{500} = -370{,}250$

37. $n = 16$

38. $S_{16} = 352$

39. $S_{20} = 690$

Section 3.3: Geometric Sequences and Series

Geometric Sequences

Let's start out with a definition:

geometric sequence: a sequence in which the next term is found by multiplying the previous term by a constant (the common ratio r)

Here are some examples of geometric sequences:

a) 9, 18, 36, 72, …

b) $12, 18, 27, \frac{81}{2}, \ldots$

c) 10, –30, 90, –270, … –196830

d) –3, –12, –48, –192, …

e) 48, –36, 27, …

The common ratios of each of these sequences, in order from a) to e), is 2, $\frac{3}{2}$, –3, 4, $-\frac{3}{4}$, respectively. Note that in each of them, we can find the common ratio r by taking **any** term and dividing it by the previous term.

Like any other sequences, geometric sequences can be finite or infinite. Example c) above is finite, as the last term is specified. The others are infinite sequences.

Example

For each of the following sequences, state whether it is arithmetic, geometric, or neither.

a) 45, 15, 5, …

b) 5, 3, 1, –1, …

c) 1, 8, 27, 64, … , 1000

d) –1, 1, –1, 1, –1, 1, …

Answer

a) Geometric, because the common ratio r is $\frac{1}{3}$.

b) Arithmetic, because the common difference d is –2.

c) Neither, because there isn't either a common difference or ratio between terms. (In fact, the pattern is that $a_n = n^3$.)

d) Geometric, because the common ratio r is –1.

Again, you can define a geometric sequence in one of three ways: by listing the terms, by giving a recursive definition, or by giving a general definition.

Recursive Definitions for Geometric Sequences

Let's look at an example.

Example

Give a recursive definition for the sequence 2, 10, 50, 250, …

Answer

Recall that a recursive definition has two parts: listing the first term and giving the pattern. In this case, the pattern is multiplying the previous term by $r = 5$ to get the next term. The recursive definition is therefore

$$\begin{cases} a_1 = 2 \\ a_n = 5a_{n-1} \end{cases}$$

More generally, the recursive definition for **any** geometric sequence is

$$\begin{cases} a_1 = \text{<insert value here>} \\ a_n = a_{n-1} \times r \end{cases}$$

General Formulae for Geometric Sequences

Let's examine the previous example in more detail to see if we can recognize any patterns and come up with a general formula. Rewriting each term, we get

$$\begin{array}{l} 2,\ 10,\ \ 50,\ \ \ \ 250,\ \ \ ... \\ 2,\ 2\times 5,\ 2\times 5^2,\ 2\times 5^3,\ ... \end{array}$$

So the 3rd term equals the first times 5 squared, the 4th term equals the first times 5 cubed, and the nth term will equal the first times 5 raised to the (n – 1) power. More generally, the nth term equals the first term times r raised to the (n – 1) power, namely

$$a_n = a_1 r^{n-1}$$

for all **geometric** sequences.

Example

Write a general formula for the sequence 3, 6, 12, …

Answer

This sequence is geometric with the first term 3 and common ratio 2.

$$a_n = a_1 r^{n-1}$$
$$a_n = 3\times(2)^{n-1}$$

The general formula is then that $a_n = 3\times 2^{n-1}$.

Example

What is the 20th term in the sequence in the sequence 3, 6, 12, … ?

Answer

This is the same sequence from the previous example. We may then use the formula we derived above with $n = 20$.

$$a_n = a_1 r^{n-1}$$
$$a_{20} = 3\times 2^{20-1}$$
$$a_{20} = 3\times 2^{19}$$
$$a_{20} = 1,572,864$$

The 20th term is 1,572,864, which provides a nice example for how fast geometric sequences can grow, even for small values of r.

Example

Write a general formula for the sequence 8, 12, 18, 27, … ? What is the fifteenth term in this sequence? The fiftieth?

Answer

$$a_n = a_1 r^{n-1}$$
$$a_n = 8\left(\frac{3}{2}\right)^{n-1}$$
$$a_{15} = 8\left(\frac{3}{2}\right)^{14} \approx 2335.43$$
$$a_{50} = 8\left(\frac{3}{2}\right)^{49} \approx 3.40065\times 10^9$$

So the general formula is $a_n = 8\left(\frac{3}{2}\right)^{n-1}$ and the fifteenth and fiftieth terms are 2335.43 and 3.4×10^9, respectively.

Geometric Series

Recall that S_n is the sum of the first n terms of a series. Let's look at how a formula for S_n is derived.

$$S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_{n-2} + a_{n-1} + a_n$$
$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-3} + a_1 r^{n-2} + a_1 r^{n-1}$$

Let's take that last expression for S_n and multiply it by $-r$ to get

$$-rS_n = -a_1 r - a_1 r^2 - a_1 r^3 - a_1 r^4 - \ldots - a_1 r^{n-2} - a_1 r^{n-1} - a_1 r^n$$

Then if we add the rows for S_n and $-rS_n$, we get

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-3} + a_1 r^{n-2} + a_1 r^{n-1} \\ -rS_n &= -a_1 r - a_1 r^2 - a_1 r^3 - a_1 r^4 - \ldots - a_1 r^{n-2} - a_1 r^{n-1} - a_1 r^n \end{aligned}$$

$$S_n - rS_n = a_1 - a_1 r^n$$

since all of the terms in between these two (a_1 and $a_1 r^n$) will cancel. Then

$$S_n\left(1-r\right) = a_1\left(1-r^n\right)$$

and

$$S_n = \frac{a_1\left(1-r^n\right)}{\left(1-r\right)}$$

The last formula above is the formula for the sum of the first n terms for **any geometric series**.

Example

Find the sum of the first 20 terms of the series 3 + 6 + 12 + …

Answer

This is a geometric series with $a_1 = 3$ and $r = 2$. We want to find S_{20}.

$$S_n = \frac{a_1\left(1-r^n\right)}{\left(1-r\right)}$$

$$S_{20} = \frac{3\left(1-2^{20}\right)}{\left(1-2\right)} = 3{,}145{,}725$$

The sum of the first 20 terms is 3,145,725.

Example

Find the sum of the first forty terms of the series 8 – 12 + 18 – 27 ….

Answer

This is a geometric series with $a_1 = 8$ and $r = -\frac{3}{2}$. We want to find S_{40}.

$$S_n = \frac{a_1\left(1-r^n\right)}{\left(1-r\right)}$$

$$S_{20} = \frac{8\left(1-\left(-1.5\right)^{40}\right)}{\left(1-\left(-1.5\right)\right)} = -3.53835\times 10^7$$

The sum of the first forty terms is -3.54×10^7.

Sum of an Infinite Geometric Series

Let's take a look at the infinite series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$

What happens when we try to evaluate this sum using the S_n formula? We can put $a_1 = ½$, $r = ½$, and $n = \infty$ into the formula, but we will run into a roadblock when we try to evaluate $(½)^{\infty}$.

Let's take a closer look at the behaviour of $(½)^n$ for large values of n. As n gets larger, the fraction $\left(\frac{1}{2}\right)^n = \frac{1}{2^n}$ gets ever smaller. In fact, as n approaches ∞, $(½)^n$ will approach zero.

This is true for any r provided that $|r| < 1$. (If you're not familiar with the absolute value bars, $|x|$, an equivalent expression is that $-1 < r < 1$.)

Recalling that $S_n = \frac{a_1\left(1-r^n\right)}{\left(1-r\right)}$ and letting the r^n term go to zero, then

$$S_\infty = \frac{a_1}{1-r} \text{ for } -1 < r < 1$$

for any **infinite geometric series**, provided that r meets the restriction above.

Let's now revisit the series that started this discussion, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$, and evaluate it in the following example.

Example

Evaluate $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$.

Answer

This series is geometric with $a_1 = ½$ and $r = ½$. Then

$$S_\infty = \frac{a_1}{1-r} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$$

The sum of this series is 1.

Example

Evaluate $24 + 16 + \frac{32}{3} + \ldots$

Answer

This series is geometric with $a_1 = 24$ and $r = \frac{2}{3}$.

$$S_\infty = \frac{a_1}{1-r} = \frac{24}{1-\frac{2}{3}} = \frac{24}{\frac{1}{3}} = 24 \times \frac{3}{1} = 72$$

Example

Evaluate $24 - 16 + \frac{32}{3} + \ldots$

Answer

This series is identical to the previous one except that r is now negative: $a_1 = 24$ and $r = -\frac{2}{3}$.

$$S_\infty = \frac{a_1}{1-r} = \frac{24}{1-\left(-\frac{2}{3}\right)} = \frac{24}{1+\frac{2}{3}} = \frac{24}{\frac{5}{3}} = 24 \times \frac{3}{5} = \frac{72}{5} = 14.4$$

Example

Evaluate $12 + 18 + 27 + \ldots$

Answer

This series is geometric with $a_1 = 12$ and $r = \frac{3}{2}$. You may already realize what's going on, but in case you don't, let's naively put the values into the formula and see what we get:

$$S_\infty = \frac{a_1}{1-r} = \frac{12}{1-\frac{3}{2}} = \frac{12}{-\frac{1}{2}} = 12 \times -\frac{2}{1} = -24$$

Wait! How can the sum of a bunch of positive number be negative? The answer is that our restriction for r is that it must be between –1 and 1, but $r = 1.5$. Because r does not satisfy the restriction, we cannot use the above formula for S_∞. Indeed, if you add up a bunch of positive numbers that are increasing as you go up, you can see that the sum just keeps getting bigger as we add more terms. You could then either say that the sum is infinite (dicey) or "does not exist" (safer).

But why is it safer to say "does not exist" in the last example? Let's look at three sums:

a) $12 + 18 + 27 + \ldots$

b) –12 – 18 – 27 – …

c) 12 – 18 + 27 + …

Each term in a) is getting more positive, so the sum of that sequence will be +∞. Each term in b) is getting more and more negative, so the sum of that sequence will be –∞. But in the last term, the sum oscillates back and forth: $S_1 = 12$, $S_2 = -6$, $S_3 = 21$, $S_4 = -19.5$, and so on. The sign of S_n is either positive or negative depending on whether the number of terms you've added is even or odd. Rather than debating whether infinity is odd or even (?!), we will just say that the sum "does not exist."

Example

Evaluate $\sum_{j=0}^{\infty} 27\left(\frac{1}{3}\right)^j$.

Answer

Ick! The best place to start is to figure out the first few terms to determine the pattern:

when $j = 0$, $27\left(\frac{1}{3}\right)^0 = 27 \times 1 = 27$

when $j = 1$, $27\left(\frac{1}{3}\right)^1 = 27 \times \frac{1}{3} = 9$

when $j = 2$, $27\left(\frac{1}{3}\right)^2 = 27 \times \frac{1}{3^2} = 3$

so our sequence is 27, 9, 3, … This is geometric with $a_1 = 27$ and $r = \frac{1}{3}$. Then

$$S_\infty = \frac{a_1}{1-r} = \frac{27}{1-\frac{1}{3}} = \frac{27}{\frac{2}{3}} = 27 \times \frac{3}{2} = \frac{81}{2} = 40.5$$

Example

Evaluate $\sum_{k=5}^{\infty} \frac{1}{2}k$.

Answer

Once again, let's figure out the first few terms to determine the pattern:

when $k = 5$, $\frac{1}{2}k = \frac{1}{2}5 = 2.5$

when $k = 6$, $\frac{1}{2}k = \frac{1}{2}6 = 3$

when $k = 7$, $\frac{1}{2}k = \frac{1}{2}7 = 3.5$

so our sequence is 2.5, 3, 3.5. Wait! This is arithmetic! Not only that, but the numbers are increasing. So the sum will be infinite, or if you prefer, the sum "does not exist".

Repeating Decimals

Let's examine $0.\overline{7}$ in some detail to see what we find:

$$\begin{aligned} 0.\overline{7} &= 0.777777777... \\ &= 0.7 + 0.07 + 0.007 + 0.0007 + ... \end{aligned}$$

But this is just the sum of an infinite series with $a_1 = 0.7$ and $r = 0.1$. Rewriting a_1 and r in fraction form (you'll see why in a minute) gives $a_1 = \frac{7}{10}$ and $r = \frac{1}{10}$. Then

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{10} \times \frac{10}{9} = \frac{7}{9}$$

So $0.\overline{7} = 7/9$. Interesting!

Example

Find an exact fraction for $0.\overline{6}$.

Answer

$$\begin{aligned} 0.\overline{6} &= 0.66666666... \\ &= 0.6 + 0.06 + 0.006 + 0.0006 + ... \end{aligned}$$

But this is just the sum of an infinite series with $a_1 = \frac{6}{10}$ and $r = \frac{1}{10}$. Then

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{6}{10}}{1-\frac{1}{10}} = \frac{\frac{6}{10}}{\frac{9}{10}} = \frac{6}{10} \times \frac{10}{9} = \frac{6}{9} = \frac{2}{3}$$

So $0.\overline{6} = 2/3$.

Example

Find an exact fraction for $0.\overline{18}$.

Answer

$$\begin{aligned} 0.\overline{18} &= 0.1818181818... \\ &= 0.18 + 0.0018 + 0.000018 + ... \end{aligned}$$

But this is just the sum of an infinite series with $a_1 = \frac{18}{100}$ and $r = \frac{1}{100}$. Then

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{18}{100}}{1-\frac{1}{100}} = \frac{\frac{18}{100}}{\frac{90}{100}} = \frac{18}{100} \times \frac{100}{99} = \frac{18}{99} = \frac{2}{11}$$

So $0.\overline{18} = 2/11$.

Summary

For a **geometric** sequence, the *n*th term is given by $a_n = a_1 r^{n-1}$

For a **geometric** series, the sum of the first *n* terms (*n*th partial sum) is $S_n = \frac{a_1(1-r^n)}{(1-r)}$

For an **infinite geometric** series, the sum is $S_\infty = \frac{a_1}{1-r}$, provided that $-1 < r < 1$.

Section 3.3: Geometric Sequences and Series

Exercises

State whether the following sequences are geometric or not. If they are, state the first term and common ratio.

1. 8, 9, 11, 13, 16, …

2. –3, –10, –17, –24, …

3. 3, 6, 12, 24, …

4. 1, 2, 6, 24, …

5. 81, 72, 63, 54, …

6. $72, 48, 32, ...$

Give both the general formula (or n[th] term a_n) and the recursive formula for the following geometric sequences.

7. 1, 3, 9, 27, …

8. 64, 16, 4, 1, …

9. 2, –6, 12, –24, …

10. 24, 2.4, 0.24, …

For the following sequences, calculate a_{50} and a_{261}.

11. 12, 18, 27,…

12. 12, 8, $\frac{16}{3}$, …

State whether the following recursively defined sequences are geometric or not. (Is there an easy way to tell?)

13. $\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + 4 \end{cases}$

14. $\begin{cases} a_1 = 12 \\ a_n = 2a_{n-1} \end{cases}$

15. $\begin{cases} a_1 = 75 \\ a_n = 10a_{n-1} \end{cases}$

16. $\begin{cases} a_1 = 7 \\ a_n = 2 - a_{n-1} \end{cases}$

17. $\begin{cases} a_1 = 8 \\ a_n = -a_{n-1} \end{cases}$

18. $\begin{cases} a_1 = 3 \\ a_n = (a_{n-1})^2 \end{cases}$

19. For the following sequence, calculate the 201st term: 5, 15, 45, …

20. For the following sequence, calculate the 20th term: 7, –14, 28, …

21. Calculate S_{20} for the series 100 + 50 + 25 + …

22. Calculate S_{20} for the series 100 + 200 + 400 + …

Calculate the sum, if it exists, for the following series.

23. $-6 + 4 - \frac{8}{3} + ...$

24. 100 + 50 + 25 + …

25. 100 + 200 + 400 + …

26. 12 + 3 + $\frac{3}{4}$ + …

Calculate the following sums, if they exist.

27. $\sum_{k=0}^{10} 2^{k+2}$

28. $\sum_{j=1}^{\infty} 15\left(\frac{3}{5}\right)^j$

29. $\sum_{i=2}^{\infty} 25(0.1)^j$

30. $\sum_{i=0}^{\infty} 4(-3)^i$

31. If the number of vampires in Transylvania doubles every month, then how many vampires will be in Transylvania in 3 years, starting from one individual? Comment on your result if the total population of Transylvania is 2 million people.

32. As I was going to St. Ives, I met a man with seven wives. Each wife had seven sacks. Each sack had seven cats. Each sack had seven kits. Kits, cats, sacks, wives: does this form a geometric sequence?

33. The paper used in the photocopier by Pat's office is said to be 0.097 mm thick. If it is folded over repeatedly, doubling its thickness each time, how thick will the paper be if it's folded 7 times? Bonus: why, then, were the Mythbusters having so many problems trying to fold the paper this many times?

Section 3.3: Geometric Sequences and Series

Solutions

1. no
2. no
3. yes, $r = 2$
4. no
5. no
6. yes, $r = \frac{2}{3}$
7. $a_n = (3)^{n-1}$ and $\begin{cases} a_1 = 1 \\ a_n = 3a_{n-1} \end{cases}$
8. $a_n = 64\left(\frac{1}{4}\right)^{n-1}$ and $\begin{cases} a_1 = 64 \\ a_n = \frac{a_{n-1}}{4} \end{cases}$
9. $a_n = 2(-3)^{n-1}$ and $\begin{cases} a_1 = 2 \\ a_n = -3a_{n-1} \end{cases}$
10. $a_n = 24(0.1)^{n-1}$ and $\begin{cases} a_1 = 24 \\ a_n = 0.1 \times a_{n-1} \end{cases}$
11. $a_n = 12\left(\frac{3}{2}\right)^{n-1}$, so $a_{50} \approx 5.1 \times 10^9$ and $a_{261} \approx 7.3 \times 10^{46}$
12. $a_n = 12\left(\frac{2}{3}\right)^{n-1}$, so $a_{50} \approx 2.8 \times 10^{-8}$ and $a_{261} \approx 1.97 \times 10^{-45}$
13. no
14. yes, with $r = 2$
15. yes, with $r = 10$
16. no

17. yes, with $r = -1$

18. no

19. $a_n = 5(3)^{n-1}$, so $a_{201} = 5(3)^{200} = 1.33 \times 10^{96}$

20. $a_n = 7(-2)^{n-1}$, so $a_{20} = 7(-2)^{19} = -3{,}670{,}016$

21. $S_{20} = 200$ (The exact answer is $\frac{26214375}{131072}$ or 1.99980926513671875, but if you round to three decimals, the answer is 200.000.)

22. $S_{20} = 104{,}857{,}500$

23. $S_\infty = \frac{a_1}{1-r} = \frac{-6}{1-(-2/3)} = -\frac{18}{5} = -3.6$

24. $S_\infty = 200$

25. S_∞ does not exist ($r > 1$)

26. $S_\infty = 16$

27.
$$\begin{aligned} S_{11} &= 2^2 + 2^3 + 2^4 + \ldots + 2^{12} \\ &= \frac{a_1(1-r^n)}{1-r} = \frac{2^2(1-2^{11})}{1-2} \\ &= 8188 \end{aligned}$$

28. $S_\infty = 22.5$

29. $S_\infty = \frac{5}{18} = 0.2\overline{7}$

30. S_∞ does not exist ($r < -1$)

31. 3 years is 36 months, so we have a 36-term sequence starting with 1, 2, 4, 8, … The nth term will be $a_n = 1(2)^{n-1}$, so the 36th term will be $a_{36} = 1(2)^{35} = 34{,}359{,}738{,}368$, which is a tad larger than the total population of Transylvania.

32. 1 man

7 wives

sacks = #wives × # sacks/wife = 7 × 7 = 49

cats = #sacks × # cats/sack = 49 × 7 = 343

kits = #cats × # kits/cat = 343 × 7 = 2401

So kits, cats, sacks, and wives is 2401, 373, 49, 7, which is a geometric sequence with four terms: $a_1 = 2401$ and $r = 1/7$.

33. After one fold, the thickness will be 0.097×2, after two folds 0.097×2^2, etc. So our starting term will be 0.097×2 and then will double with $r = 2$ thereafter.

So $a_n = 0.097(2)^n$, and the 7th term will be $a_7 = 0.097(2)^7 = 12.416$, and the paper thickness will be 12.4 mm, or just over 1 cm thick.

(The Mythbusters realized that the problems with paperfolding lie with the fold itself. If I remember correctly, they resorted to C-clamps and hitting the fold with a hammer to flatten it.)