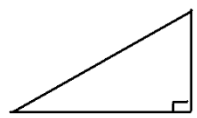
Section 4.1: Introduction to Trigonometry

Review of Triangles

Recall that the sum of all angles in any triangle is 180°. Let's look at what this means for a right triangle:

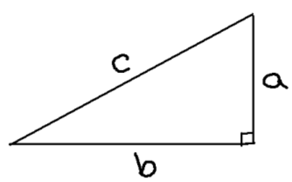


A right angle is an angle which measures 90°. A right triangle, then, has one 90° angle, and since the sum of all angles must be 180°, the other two angles in the triangle must add to 90°. In the diagram, the right angle is denoted by a little rectangle at the vertex (corner) of the right angle.

Since the other two angles sum to 90°, they must both be acute (less than 90°). Angles between 90° and 180° are called obtuse.

Pythagorean Theorem

The Pythagorean Theorem states that if a triangle is a right triangle, then $a^2 + b^2 = c^2$ (as shown in the diagram) and vice versa.

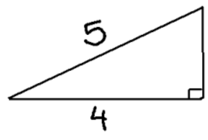


In the diagram we use the naming convention for right triangles, in which c is the hypotenuse (the side opposite the right angle) and a and b are the other two sides of the triangle in any order.

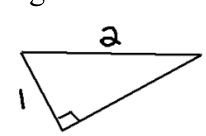
Example

Calculate the remaining side for each of the following triangles.

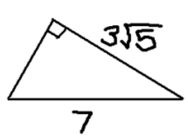
a)



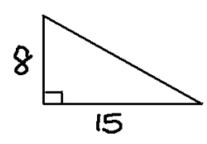
b)



c)



d)



Answer:

a) Let's call the remaining side *a*. Then *b* = 4, *c* = 5, and

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 4^2 &= 5^2 \\ a^2 + 16 &= 25 \\ a^2 &= 9 \\ a &= 3 \end{aligned}$$

To be rigorous, we really should say that *a* could be ±3, but triangles can't have negative sides, so we only take the positive answer, 3.

b) Let's call the remaining side *b*. Then *a* = 1, *c* = 2, and

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 1^2 + b^2 &= 2^2 \\ 1 + b^2 &= 4 \\ b^2 &= 3 \\ b &= \sqrt{3} \end{aligned}$$

c) Let's call the remaining side *a*. Then *b* = $3\sqrt{5}$, *c* = 7, and

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + \left(3\sqrt{5}\right)^2 &= 7^2 \\ a^2 + 45 &= 49 \\ a^2 &= 4 \\ a &= 2 \end{aligned}$$

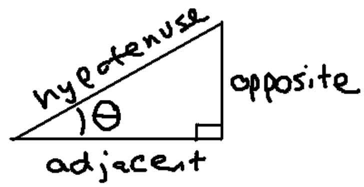
(Remember that $\left(3\sqrt{5}\right)^2 = \left(3\sqrt{5}\right)\left(3\sqrt{5}\right) = 3^2\left(\sqrt{5}\right)^2 = 9 \times 5 = 45$.)

d) The remaining side is the hypotenuse, *c*. Then *a* = 8, *b* = 15, and

$$\begin{aligned} c^2 &= a^2 + b^2 \\ c^2 &= 8^2 + 15^2 = 64 + 225 = 289 \\ c &= 17 \end{aligned}$$

Trigonometric Ratios

Let's consider a right triangle as shown in the diagram below. We've labeled one of the two acute angles with the variable name θ, which is the Greek letter "theta". There is a naming convention, then, for the remaining two sides of the triangle. The longest side is still called the hypotenuse, but now the side opposite to the angle θ is called the "opposite", while the side next to the θ is called the "adjacent" side.



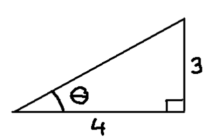
The trig ratios are then just ratios of these sides. The three basic trig functions, which are the ones we will study in this course, are called **sine**, **cosine**, and **tangent**. They are defined as follows.

$$\sin\theta = \frac{opposite}{hypotenuse}$$
$$\cos\theta = \frac{adjacent}{hypotenuse}$$
$$\tan\theta = \frac{opposite}{adjacent}$$

The easy way to remember them is to take the first letter of each word: SOHCAHTOA. Then the SOH stands for Sin/Opp/Hyp, CAH for Cos/Adj/Hyp, and TOA for Tan/Opp/Adj.

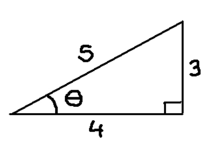
Example

Calculate the three basic trig functions of θ exactly for the following triangle.



Answer:

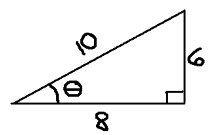
First, we need to calculate the hypotenuse. We can either note that it's the same 3-4-5 triangle as in the first example, or use the Pythagorean theorem to calculate that $c = 5$.



Then we note that for this θ, the opposite side is 3 and the adjacent side is 4. So,

$$\sin\theta = \frac{opp}{hyp} = \frac{3}{5}$$
$$\cos\theta = \frac{adj}{hyp} = \frac{4}{5}$$
$$\tan\theta = \frac{opp}{adj} = \frac{3}{4}$$

What happens if we look at the following triangle?



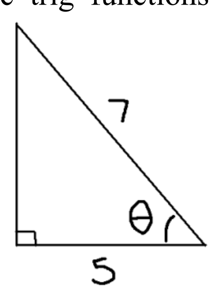
Since this is just the 3-4-5 triangle scaled up or enlarged by a factor of 2, the two triangles are similar, so the two θs are equal to each other. Let's look at what the trig ratios would be for the θ in the larger triangle.

$$\sin\theta = \frac{opp}{hyp} = \frac{6}{10} = \frac{3}{5}$$
$$\cos\theta = \frac{adj}{hyp} = \frac{8}{10} = \frac{4}{5}$$
$$\tan\theta = \frac{opp}{adj} = \frac{6}{8} = \frac{3}{4}$$

And you'll notice that the trig ratios are the same as for the previous triangle. What that means is that the **trig ratios are only a function of the angle θ** and do not depend on the size of the triangle. After all, if you scale the triangle up by 2, then all three sides increase by the same factor. When you take the ratio, that scale factor will be in both the numerator and denominator of the fraction, so will cancel.

Example

Calculate the three basic trig functions of θ exactly for the following triangle.



Answer:

First, we need to calculate the remaining side, using the Pythagorean theorem. Let's call it a. Then

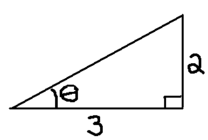
$$a^2 + b^2 = c^2$$
$$a^2 + 5^2 = 7^2$$
$$a^2 + 25 = 49$$
$$a^2 = 24$$
$$a = \sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}$$

Then we note that for this θ, the opposite side is $2\sqrt{6}$ and the adjacent side is 5. Therefore,

$$\sin\theta = \frac{opp}{hyp} = \frac{2\sqrt{6}}{7}$$
$$\cos\theta = \frac{adj}{hyp} = \frac{5}{7}$$
$$\tan\theta = \frac{opp}{adj} = \frac{2\sqrt{6}}{5}$$

Example

Calculate the three basic trig functions of θ exactly for the following triangle.



Answer:

First, we need to calculate the hypotenuse, using the Pythagorean theorem.

$$c^2 = a^2 + b^2$$
$$c^2 = 2^2 + 3^2 = 4 + 9 = 13$$
$$c = \sqrt{13}$$

Then we note that for this θ, the opposite side is 2 and the adjacent side is 3. So,

$$\sin\theta = \frac{opp}{hyp} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

$$\cos\theta = \frac{adj}{hyp} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\tan\theta = \frac{opp}{adj} = \frac{2}{3}$$

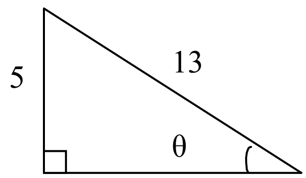
Remember that you have to rationalize the denominator when you are simplifying fractions containing radicals!

Section 4.1: Introduction to Trigonometry

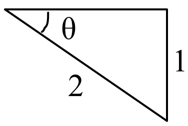
Exercises

Calculate the remaining side for the following right triangles. Give exact answers.

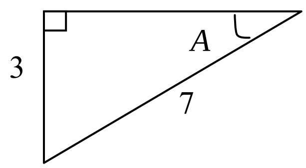
1.



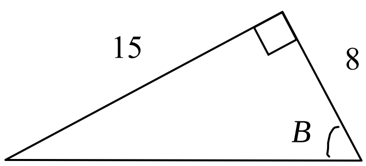
2.



3.



4.



5. $a = 2, b = 3$

6. $a = \sqrt{15}, c = 4$

7. $b = 1, c = 5$

8. $a = 6, b = 8$

9. $a = 3\sqrt{5},\ b = \sqrt{5}$

10. $b = \sqrt{2},\ c = 4$

For the following right triangles, calculate all three basic trig functions of angle A exactly.

11. $a = 1,\ b = 1$

12. $a = 2\sqrt{3},\ c = 4$

13. $b = 4,\ c = 6$

14. $a = 4,\ b = 5$

Use your calculator to calculate the values of the following trig functions. Round your answer to four decimal places.

15. sin 64°

16. cos 35°

17. tan 27.5°

18. tan 89.273°

19. cos 4.3°

20. sin 4.3°

Given the following function values, calculate the value of θ if θ is acute. Round your answer to two decimal places.

21. sin θ = 0.35

22. cos θ = 0.99

23. tan θ = 103

24. sin θ = 2.1

25. cos θ = 1.6

26. tan θ = 1.6

27. (trickier) If θ is acute and sin θ = cos θ, find θ.

28. (trickier) For any right triangle, what is the relationship between sin A and cos B?

29. (irritating) For any right triangle, find a formula for the three trig functions of A in terms of *a* and *c* only.

Section 4.1: Introduction to Trigonometry

Answers

1. 12

2. $\sqrt{3}$

3. $2\sqrt{10}$

4. 17

5. $\sqrt{13}$

6. 1

7. $2\sqrt{6}$

8. 10

9. $5\sqrt{2}$

10. $\sqrt{14}$

11. $c = \sqrt{2}$, so $\sin A = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, $\cos A = \textit{same}$, $\tan A = \frac{1}{1} = 1$

12. $b = 2$, so $\sin A = \frac{\sqrt{3}}{2}$, $\cos A = \frac{1}{2}$, $\tan A = \sqrt{3}$

13. $a = 2\sqrt{5}$, so $\sin A = \frac{\sqrt{5}}{3}$, $\cos A = \frac{2}{3}$, $\tan A = \frac{\sqrt{5}}{2}$

14. $c = \sqrt{41}$, so $\sin A = \frac{4\sqrt{41}}{41}$, $\cos A = \frac{5\sqrt{41}}{41}$, $\tan A = \frac{4}{5}$

15. 0.8988

16. 0.8192

17. 0.5206

18. 78.8070

19. 0.9972

20. 0.0750

21. $\theta = 20.49°$

22. $\theta = 8.11°$

23. $\theta = 89.44°$

24. θ = undefined

25. θ = undefined

26. $\theta = 57.99°$

27. 45°

28. they are equal

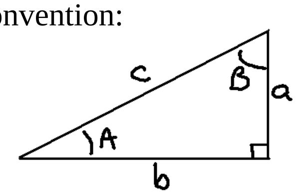
29. $\sin A = \frac{a}{c}, \quad \cos A = \frac{\sqrt{c^2 - a^2}}{c}, \quad \tan A = \frac{a}{\sqrt{c^2 - a^2}} = \frac{a\sqrt{c^2 - a^2}}{c^2 - a^2}$

Section 4.2: Applications of Right Triangles

Solving Triangles

In order to complete solve a triangle, you need three pieces of information, one of which must be a side. If we apply this to a right triangle, recall that we must then already know one angle (the right angle). We then need only two further pieces of information to solve the triangle, but as before, one of those pieces must be a side.

To **completely solve a right triangle**, then, means to take those pieces of information and calculate all remaining sides and angles. Before we do that, however, we need to look at one more naming convention:



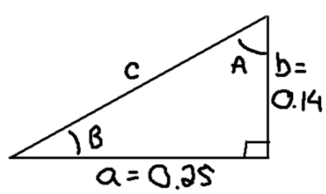
As before, the hypotenuse is *c* and the two other sides are *a* and *b*. The two acute angles are called *A* and *B*, and angle *A* is located opposite side *a* and so on. Side *c*, being the hypotenuse, must always be opposite the right angle.

Example

Solve the right triangle with $a = 0.25$ and $b = 0.14$. Round any approximate answers to two decimal places.

Answer

The diagram is shown below.



To get *c*, we use the Pythagorean theorem:

$$c^2 = a^2 + b^2$$
$$c^2 = (0.25)^2 + (0.14)^2$$
$$c = 0.286531 = 0.29$$

Then we use a ratio of sides to determine one of the two unknown angles. At this point, because all sides are known, any of the three basic trig functions may be used. I will choose to use the tangent, since that will require only the original information given in the problem. Please note that if you use your value of *c* in further calculations, it's good practice to take a few extra decimal places (not just the 0.29 value) in order to minimize round-off errors.

$$\tan B = \frac{b}{a}$$
$$B = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{0.14}{0.25}\right) = 29.2488^\circ = 29.25^\circ$$

Then we can just subtract *B* from 90° in order to determine *A*:

$$A = 90^\circ - B = 90^\circ - 29.25^\circ = 60.75^\circ$$

So the solution to the triangle is

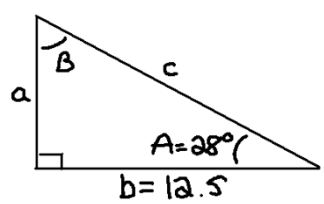
$$A = 60.75^\circ, B = 29.25^\circ, c = 0.29$$

Example

Solve the right triangle with *b* =12.5 and *A* = 28°. Round any approximate answers to one decimal place.

Answer:

The diagram is sketched below.



Firstly, we can find *B* by subtracting *A* from 90°:

$$B = 90^\circ - A = 90^\circ - 28^\circ = 62^\circ$$

We cannot initially use the Pythagorean theorem to determine another side because we only have one of the three sides to begin with. Therefore, we will have to start with a trig ratio and solve for an unknown side. If I want to calculate *c* from the information given, then

$$\cos A = \frac{b}{c}$$
$$c \cos A = b$$
$$c = \frac{b}{\cos A} = \frac{12.5}{\cos 28^\circ} = 14.1571 = 14.2$$

Similarly,

$$\tan A = \frac{a}{b}$$
$$b \tan A = a$$
$$a = b \tan A = 12.5 \tan 28^\circ = 6.64637 = 6.6$$

So the solution to the triangle is

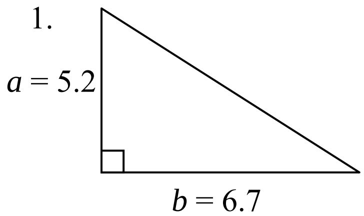
$$a = 6.6, c = 14.2, B = 62^\circ$$

Section 4.2: Applications of Right Triangles

Exercises

Solve the following right triangles, rounding approximate answers to one decimal place.

1.



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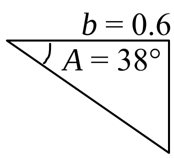

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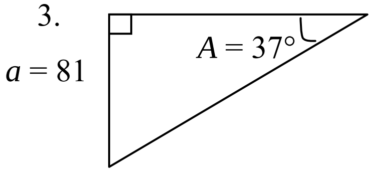

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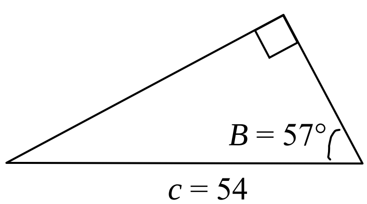

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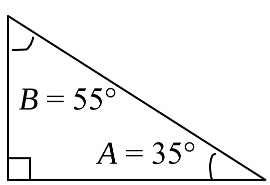
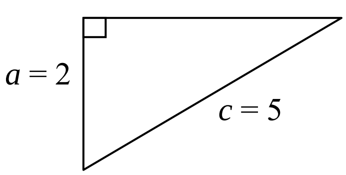


6.



7. $a = 18, b = 4$

8. $a = 103,\ c = 120$

9. $b = 0.52,\ c = 0.84$

10. $A = 21^\circ,\ b = 0.59$

11. $A = 53^\circ,\ c = 26$

12. $B = 5^\circ,\ b = 42$

13. $A = 75^\circ,\ a = 72.3$

14. $B = 30^\circ,\ a = 12$

15. $B = 62^\circ,\ c = 18.6$

16. $A = 30^\circ, B = 60^\circ$

Section 4.2: Applications of Right Triangles

Answers

1. $c = 8.5, A = 37.8^\circ, B = 52.2^\circ$

2. $a = 0.5, c = 0.8, B = 52^\circ$

3. $b = 107.5, c = 134.6, B = 53^\circ$

4. $a = 29.4, b = 45.3, A = 33^\circ$

5. cannot be solved, since don't have any sides

6. $b = 4.6, A = 23.6^\circ, B = 66.4^\circ$

7. $c = 18.4, A = 77.5^\circ, B = 12.5^\circ$

8. $b = 61.6, A = 59.1^\circ, B = 30.9^\circ$

9. $a = 0.7, A = 51.8^\circ, B = 38.2^\circ$

10. $a = 0.2, c = 0.6, B = 69^\circ$

11. $a = 20.8, b = 15.6, B = 37^\circ$

12. $a = 480.1, c = 481.9, A = 85^\circ$

13. $b = 19.4, c = 74.9, B = 15^\circ$

14. $b = 6.9, c = 13.9, A = 60^\circ$

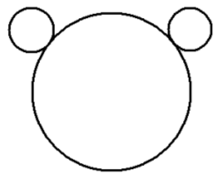
15. $a = 8.7, b = 16.4, A = 28^\circ$

16. cannot be solved, since don't have any sides

Section 4.3: Applications in Computer Graphics

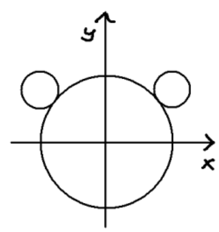
Finding coordinate points

Suppose you were designing a web page and wanted to make a little graphic of, say, Mickey Mouse or Winnie the Pooh, or your favourite cartoon mouse/bear. You might start out with a sketch that looks like the following.

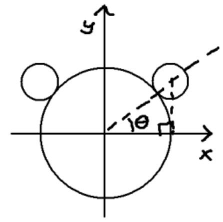


How you'll actually implement this depends on what computer program you are using. Let's work on some tools so that no matter what input is required, you can determine from your diagram any pieces of information that the software might require.

The first thing you might want to do is to impose a coordinate system onto your drawing. You may choose, as we've done here, to put the origin in the middle of your diagram. Another popular choice is to put the origin in the lower left corner. Which you'll use depends again on the software you're using. For the purposes of this course, it doesn't matter which system you pick.

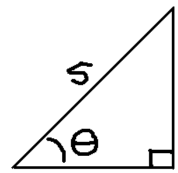


Now, the coordinates of the big circle will be (0,0) because of my choice of origin. However, we'll need to do a bit of work to determine the coordinates of the centres of the two smaller circles.

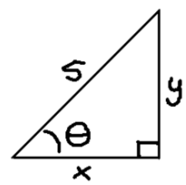


As shown in the diagram above, you can make a right triangle by drawing a line from the origin to the centre of the right-hand small circle. Then drop a line straight down to the *x*-axis.

The hypotenuse of your triangle is then just the sum of the radii of your big and little circle and the angle between the hypotenuse and the *x*-axis will be θ. Let's say that we'd like the big circle to have radius 4 and the little ones to have radius 1, in whatever units are appropriate for the software you are using. Then your triangle will look like:



Suppose you decide that you want the ears at a 45° angle, so that θ = 45°. You will have the two pieces of information about your right triangle (in addition to the right angle) that you will need to completely solve the triangle. Let's call the remaining sides *x* and *y*.



Then you'll find that

$$\sin\theta = \frac{y}{5}$$
$$5\sin\theta = y$$
$$y = 5\sin\theta = 5\sin 45^\circ \approx 3.54$$

and

$$\cos\theta = \frac{x}{5}$$
$$5\cos\theta = x$$
$$x = 5\cos\theta = 5\cos 45^\circ \approx 3.54$$

So if you chose 4 and 1 as the radii of the big and little circles, respectively, and 45° for your angle, then the coordinates of the centres of the circles will be (0,0) for the big one and (3.54, 3.54) and (−3.54, 3.54) for the two little ones.

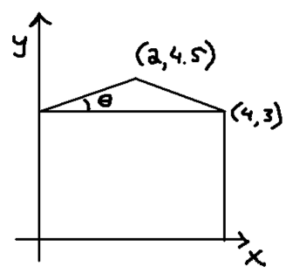
Now, for this particular setup, it's true that the *x*- and *y*-components will be the same, since we've chosen θ to be 45°. But what's nice about the analysis above is that it will still be true for any value of θ that you might wish to use, so you could experiment with a few different values and find the one that makes your diagram look like whatever you had in mind.

In fact, you could even leave your hypotenuse in terms of the two radii, $c = r_{\text{big circle}} + r_{\text{little circle}}$, and experiment with different values of the radii as well.

Let's look at another example.

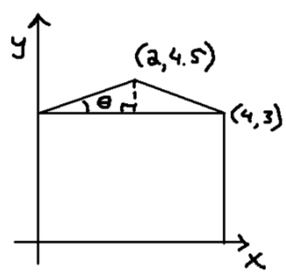
Example

Calculate the angle θ and the length of each sloping piece of roof for the house in the following diagram.

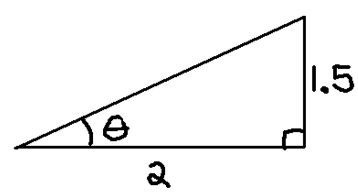


Answer:

First, let's draw a right triangle by dropping a line from the peak of the roof downwards as shown in the diagram.



By inspection, the triangle will have sides 1.5 and 2 as shown below. ("By inspection" means essentially "by looking at the diagram".)



Then we can use that

$$\tan\theta = \frac{1.5}{2}$$

$$\theta = \tan^{-1}\left(\frac{1.5}{2}\right) = 36.87^\circ = 37^\circ$$

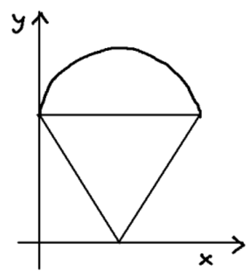
And we can calculate the remaining side using the Pythagorean theorem:

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (1.5)^2 + (2)^2 \\ c &= 2.5 \end{aligned}$$

So the angle of the roof is 37° and each piece of the sloping roof is 2.5 units long.

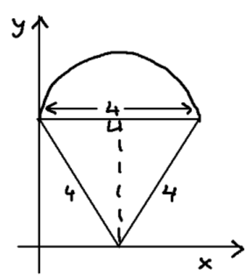
Example

A signmaker wishes to make a sign in the shape of an icecream cone, as shown in the diagram. The "cone" consists of a semicircle and an equilateral triangle. If the triangle has sides of length 4 units, calculate the coordinates of each vertex (corner) of the triangle and the radius of the semi-circle. Round, when appropriate, to two decimal places.

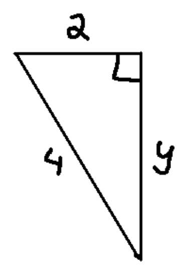


Answer:

Draw a right triangle in the diagram as shown below.



Since the figure is symmetrical, then the triangle has sides 2 and 4 as shown below.



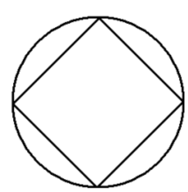
Using the Pythagorean theorem to calculate y:

$$a^2 + b^2 = c^2$$
$$4 + y^2 = 16$$
$$y^2 = 12$$
$$y = 2\sqrt{3} \approx 3.46$$

So the coordinates of the three vertices are (2, 0), (0, 3.46), and (4, 3.46), and the semicircle has a radius of 2 units.

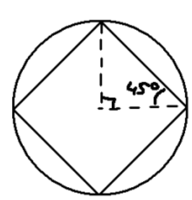
Example

A graphic is made from a square inscribed inside a circle as shown below. If the circle has a radius of 5 cm, calculate the length of the side of the square. Round your answer to two decimal places.

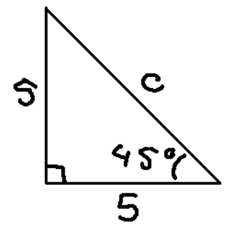


Answer

Draw a right triangle as shown below. (You could also continue the straight line down so that the diameter of the circle is the hypotenuse, but either triangle will do.)



Then two of the sides will have the same length as the radius of the circle, like so:



And then *c* can be found by Pythagorus (or trig, if you prefer):

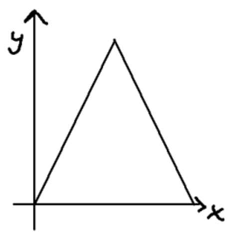
$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 25 + 25 \\ &= 50 \\ c &= \sqrt{50} = 5\sqrt{2} \approx 7.07 \end{aligned}$$

and the side of the inscribed square is 7.07 cm long.

Section 4.3: Applications to Computer Graphics

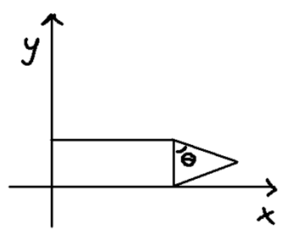
Exercises

1. Consider the triangle in the diagram below. If the base is 3 units long and the two sloping sides are each 5 units long, calculate the coordinates of each vertex (corner).

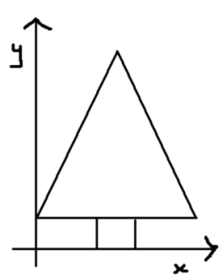


2. Use the same diagram as question 1, above. If the coordinates of the three vertices are (0,0), (3,0) and (1.8, 3.5), calculate the lengths of the three sides and the sizes of the three angles and mark them in the appropriate places on the diagram.

3. A Crayola crayon manufacturer wants to make a sign in the form of a crayon, as shown in the diagram below. The rectangle is 4 units long and 2 units wide. The vertical side of the triangle is also 2 units long, while the other two sides are both 3 units long. Calculate the coordinates of the pointed end of the crayon and the angle θ in the diagram.

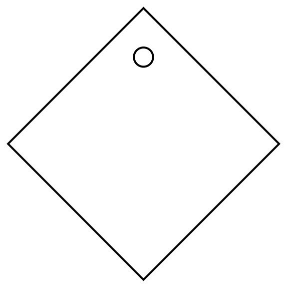


4. Suppose you wanted to make a sign in the shape of a tree, as shown below. The trunk is a square of side 2 units, while the base of the triangle is 6 units long. If the triangle is isosceles (the two sloped sides are equal in length) and the angle between the horizontal base and one of the sloped sides is 65°, what are the coordinates of the three vertices of the triangle?

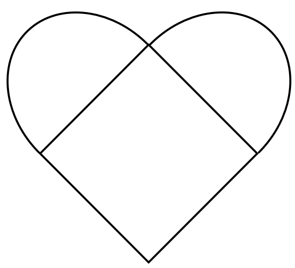


5. A pendant in the shape of a square is shown below. If the centre of the circle to be drilled in the pendent is 1 unit below the top of the pendent and the pendent has side

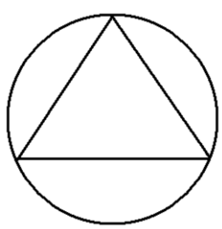
of length 5 units, how far is the centre of the circle from the bottom point of the pendant?



6. A computer graphic in the shape of a heart is shown below. It consists of two semicircles and a square. If the square has side of length 12 and the origin is placed at the bottom of the square, what are the coordinates of the centres of the two semicircles?



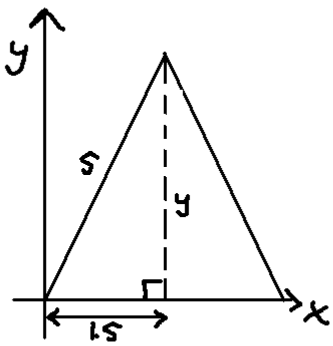
7. (tricksy) An equilateral triangle is inscribed inside a circle, as shown below. If the triangle has a side of length 2 units, calculate the radius of the circle.



Section 4.3: Applications to Computer Graphics

Solutions

1. The triangle is isosceles, since the two sloping sides are the same length. Dropping a dotted line straight down to the base from the vertex will therefore cut the base in half, making a right triangle as shown below.



We can then find y from the Pythagorean theorem:

$$a^2 + b^2 = c^2$$

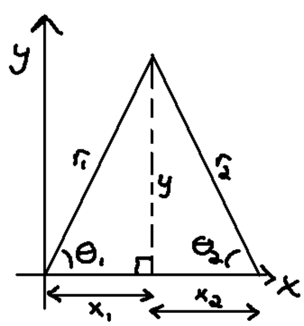
$$(1.5)^2 + y^2 = 5^2$$

$$y^2 = 5^2 - 1.5^2$$

$$y = 4.77$$

The coordinates of the vertex are then (1.5, 4.8).

2. Unlike the previous problem, this triangle is not isosceles. Then $x_1 = 1.8$ and $x_2 = 1.2$ and $y = 3.5$, from the information given in the problem.



Using the Pythagorean theorem to get the two hypotenuses:

$$c^2 = a^2 + b^2 \qquad\qquad c^2 = a^2 + b^2$$

$$r_1^2 = x_1^2 + y^2 \qquad \text{and} \qquad r_2^2 = x_2^2 + y^2$$

$$r_1^2 = (1.8)^2 + (3.5)^2 \qquad\qquad r_2^2 = (1.2)^2 + (3.5)^2$$

$$r_1 = 3.94 \qquad\qquad r_2 = 3.7$$

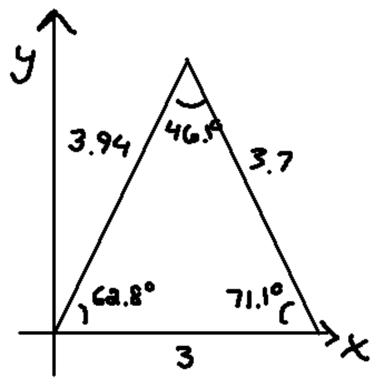
We can find the angles θ_1 and θ_2 using right-triangle trig:

$\tan\theta_1 = \frac{y}{x_1} = \frac{3.5}{1.8}$ and $\tan\theta_2 = \frac{y}{x_1} = \frac{3.5}{1.2}$

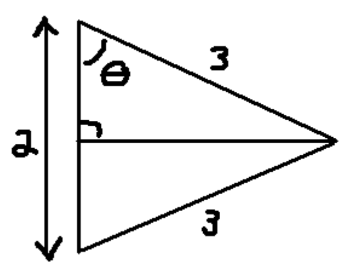
$\theta_1 = 62.8^\circ$ $\qquad\qquad$ $\theta_2 = 71.1^\circ$

The third angle at the top may be found by subtracting the sum of the other two from 180°: $\theta_3 = 180^\circ - \theta_1 - \theta_2 = 180^\circ - 62.8^\circ - 71.1^\circ = 46.1^\circ$.

Our triangle, therefore, looks like this:



3. Blowing up the triangle forming the point of the crayon, we get the following diagram.



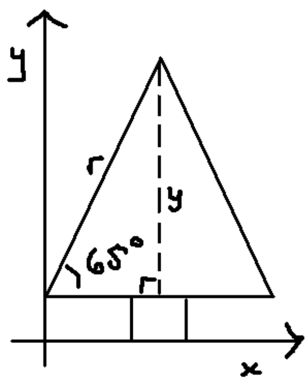
The horizontal line splits the vertical side in two, so the side adjacent to θ equals 1. Then the opposite side may be found by the Pythagorean theorem to be $2\sqrt{2}$, approximately 2.83. To calculate θ, use right-triangle trig:

$\cos\theta = \frac{adj}{hyp} = \frac{1}{3}$

$\theta = 70.5^\circ$

So the angle θ in the diagram is 70.5° and the coordinates of the point of the triangle will be (4 + 2.83, 1) = (6.83, 1).

4. Dropping a dotted line down from the top of the triangle, we get a right triangle as shown below. The length of the triangle adjacent to the 65° angle is 3, since the base of the triangle is 6 units long.



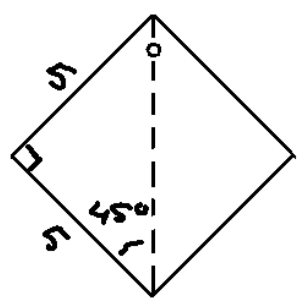
The side *y* may be found using right-triangle trig:

$$\tan 65^\circ = \frac{y}{3}$$

$$y = 3\tan 65^\circ \approx 6.43$$

The coordinates of the triangle's vertices are therefore (0,2), (6,2), and (3, 8.43).

5. We can make a right triangle with 45° angles by dropping a vertical line from the top of the pendent to the bottom. The resulting isosceles right triangle is shown in the diagram.



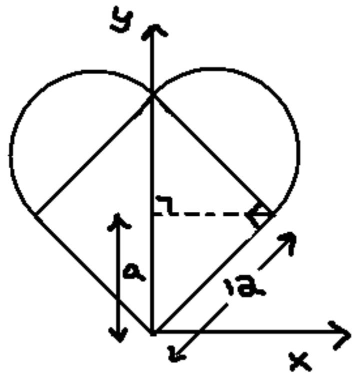
We can then find the hypotenuse by the Pythagorean theorem:

$$c^2 = a^2 + b^2 = 25 + 25$$

$$c = \sqrt{50} = 5\sqrt{2} \approx 7.07$$

The centre of the circle is therefore either $5\sqrt{2} - 1$ units or approximately 6.07 units from the bottom of the pendant.

6. Making a right triangle using one of the sides of the original square as the hypotenuse, we find that the resulting triangle is isosceles (two sides equal).



Letting this side equal *a*, then we can find its length using the Pythagorean theorem:

$a^2 + b^2 = c^2$

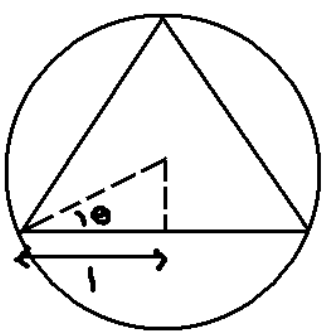
$2a^2 = 144$

$a^2 = 72$

$a = \sqrt{72} = 6\sqrt{2} \approx 8.49$

The coordinates of the centre of the right-hand semicircle can then be found in terms of *a*. The centre will be located halfway along the hypotenuse of the right triangle, so will have an *x*-coordinate equal to half of *a*, while the *y*-coordinate will be one-and-a-half times *a*. Since $a = 6\sqrt{2}$, then the centre of the right semicircle will be at $\left(3\sqrt{2}, 9\sqrt{2}\right)$, which is approximately equal to (4.24, 12.73). The left semicircle will have the opposite *x*-coordinate and the same *y*-coordinate: (–4.24, 12.73).

7. Let's make a right triangle by dropping a line from the centre of the circle/triangle to the bottom side and to the left corner of the triangle, as shown below.



The angle θ is half of the 60° angle making up one of the angles of this equilateral triangle. The hypotenuse of this triangle will be equal in length to the radius of the circle. Using the Pythagorean theorem,

$$\cos 30^\circ = \frac{1}{r}$$

$$r = \frac{1}{\cos 30^\circ} = 1.15$$

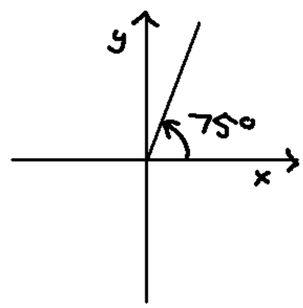
so the radius of the triangle is 1.15 units long ($\frac{2\sqrt{3}}{3}$ exactly).

Section 4.4: Trig Functions of Any Angle

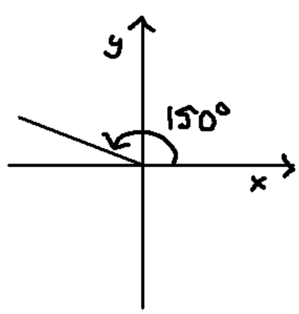
Angles in Standard Position

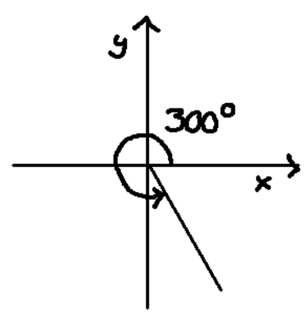
Suppose we don't want to restrict ourselves to acute angles. Let's take a look at how we can define larger angles using a coordinate plane.

An **angle in standard position** is an angle whose initial arm is on the positive *x*-axis and which rotates counter-clockwise to the final or **terminal** arm. For example, a 75° angle would be plotted as follows.

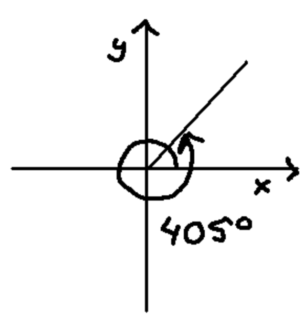


But there's no reason to restrict our angles to the first quadrant. For angles larger than 90°, we can simply keep rotating until we reach the terminal arm. To see what that looks like, let's look at the 150° and 300° angles graphed below.

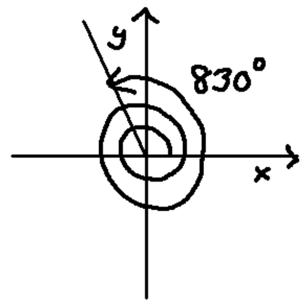




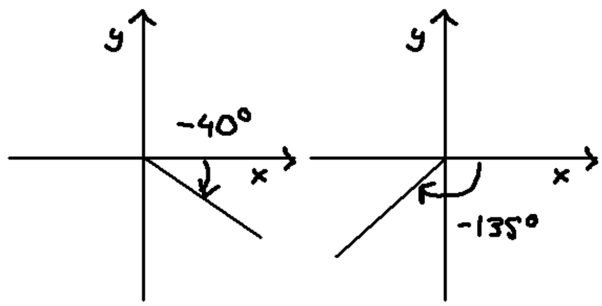
In fact, we don't have to restrict ourselves to angles less than 360°. We can just keep going, so that 405° would continue for another 45° past the *x*-axis, as in the diagram.



Similarly, 830° would be two full rotations (720°) plus another 110°, putting the terminal arm in Quadrant II.

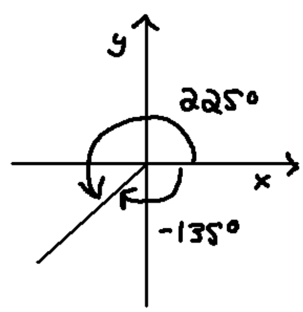


Then negative angles can be represented by **clockwise** rotations. –40° and –135° are plotted below, and there is no reason that you couldn't also have –900° and other angles requiring more than one full rotation.



Coterminal Angles

If you take a look at the diagram below, you'll notice that 225° and –135° both have the same terminal arm. Angles with the same terminal arm are called **coterminal angles**.



If you think about it, though, there's no reason that you couldn't just add another rotation to 225° to get (225° + 360°) = 585° as another coterminal angle. In fact, there are an infinite number of coterminal angles for 225°, and one way to represent this would be to say that if θ is coterminal with 225°, then

$$\theta = 225° + n \cdot 360°$$

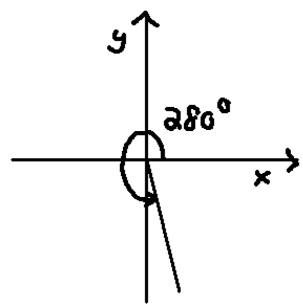
where n is an integer (if n is negative, then you are just adding counterclockwise rotations).

Example

Plot 280° in the coordinate plane, and list one positive and one negative coterminal angle.

Answer

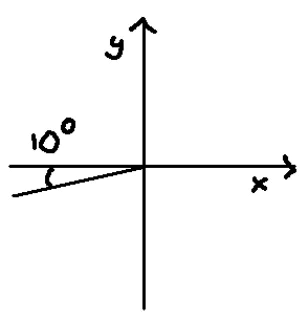
The sketch of 280° is shown below. The angle between the terminal arm and the x-axis is 80°, so –80° is a negative coterminal angle. Adding one rotation to 280° gives (280° + 360° = 640°) as a positive coterminal angle.

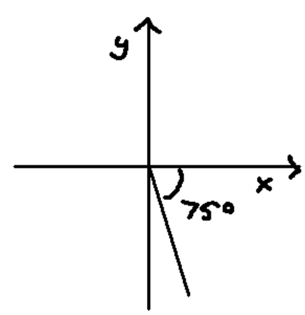


However, you don't just have to limit yourself to one rotation either way. Other acceptable answers would include (280° + 5×360° = 2080°), (280° + (–10)×360° = –3320°) and similar answers. In fact, each angle has an infinite number of coterminal angles.

Reference Angles

When θ is in standard position, its **reference angle** is the angle between the terminal arm and the nearest x-axis. Reference angles are always positive. For example, the angle 190° has a reference angle of 10° and the angle 285° has a reference angle of 75°, as shown below.



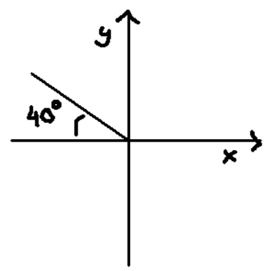


Example

What is the reference angle to –580°?

Answer

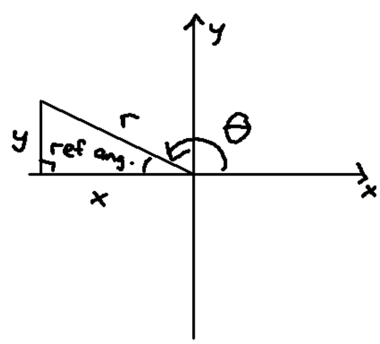
–580° is one clockwise rotation plus 220°, so will have its terminal arm in QII, as shown below. The reference angle is then the angle between the terminal arm and the negative *x*-axis, 40°.



Generalizing from Triangles to Coordinate Plane

When an angle is in standard position, then it can take on any value in the set of real numbers. What does this mean for trig functions of larger angles?

Consider the angle 150°. It lies in the second quadrant, as shown below, and has a reference angle of 30°. If we drop a line down to the closest *x*-axis to make a right angle, then we have set up a little right triangle containing the reference angle. Since we are now in a coordinate system, we can consider the sides of the triangle to now be **coordinates** instead of lengths.



If we label the adjacent side as *x* and the opposite side as *y*, then the setup looks very similar to our right-triangle trigonometry of the previous section. However, the *x*- and *y*-coordinates could take on **positive** values, unlike lengths of sides which are always positive. As you can see in the diagram above, reference angles in QII will give triangles in which the *x*-value is negative while the *y*-value is positive.

We've labeled the hypotenuse as *r* in the diagram. It is considered to be the distance from the point (*x*,*y*) to the origin, and is always positive. We can still use the Pythagorean theorem to calculate its value from *x* and *y*.

Looking at the diagram once again, then the sine of any angle θ in standard position will be

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{r}$$
$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{r}$$
$$\tan\theta = \frac{opposite}{adjacent} = \frac{y}{x}$$

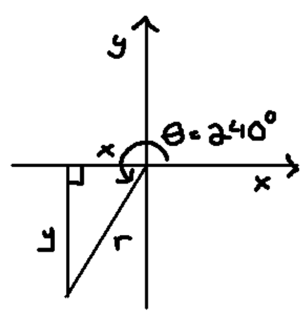
where *x* and *y* can now be any real number and *r* is a positive quantity. The resulting trig functions can then take on both positive and negative values.

Example

Will the trig functions of 240° be positive or negative?

Answer

240° is in the third quadrant, as shown below. Therefore, both *x* and *y* will be negative, making sin θ and cos θ negative as well. However, tan θ will be positive, since *y/x* will be a positive quantity.

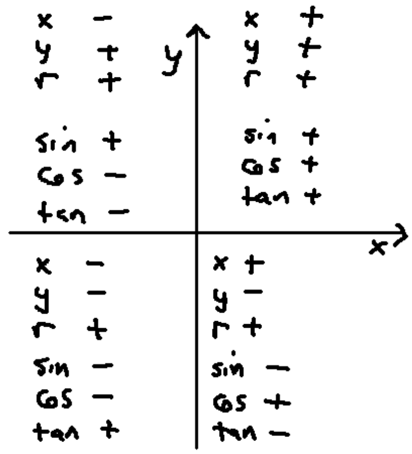


In fact, if you calculate the values of the trig functions of 240° using a calculator, you'll find that

$$\sin 240^\circ = -0.866$$
$$\cos 240^\circ = -0.5$$
$$\tan 240^\circ = 1.732$$

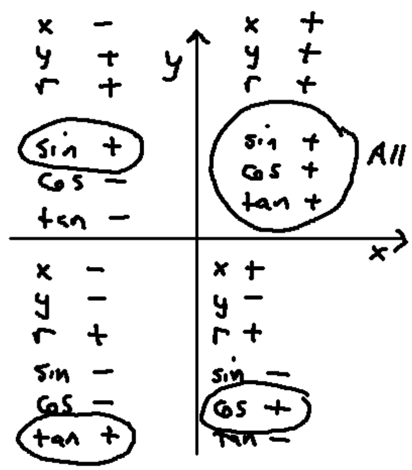
and the sign of each trig function (positive or negative) is consistent with our analysis above.

Is there a quick way to determine whether the trig functions for a particular angle will be positive or negative? Consider the following chart. In each quadrant, the sign for *x*-values and *y*-values is written in. Then the sign of each trig function is determined.

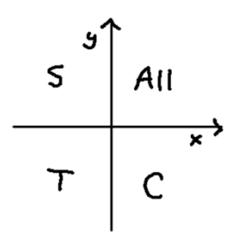


For example, in Quadrant IV, *x*-values will be positive and *y*-values will be negative, giving rise to a positive cosine (*x/r*) and negative sine (*y/r*) and tangent (*y/x*).

An easy way to remember this is to circle the trig functions in each quadrant that are positive:



Then All trig functions in Quadrant I are positive, Sine is positive in Quadrant II, Tangent in QIII, and Cosine in Quadrant IV, like so:



And this can be remembered by the phrase, "All Students Take Calculus", starting with the All in quadrant I and rotating counterclockwise, just as in angles. Other similar sayings include "All Suckers Take Chemistry", "All Saskatchewaners Tip Cows", or anything that uses ASTC that is easy to remember.

Example

State whether the three basic trig functions are positive or negative for angles in Quadrant IV.

Answer

From the "All Students Take Calculus" rule, Quadrant IV has C for cosine positive, so the sine and tangent functions will be negative there.

Example

Calculate the values of the following trig functions using a calculator. Round your answers to four decimal places.

a) $\sin 2543°$

b) $\cos -172°$

c) $\tan 270°$

d) $\sin 540°$

Answer:

a) $\sin 2543° = 0.3907$

b) $\cos -172.224° = -0.9908$

c) $\tan 270° =$ undefined

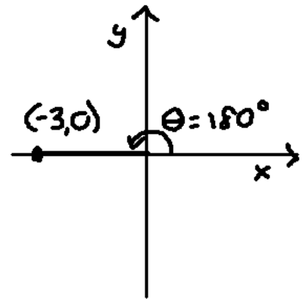
d) $\sin 540° = 0$ (exact, but you can add decimal places if you wish)

Quadrantal Angles

What if the angle you are interested lies on one of the axes? What happens to the trig functions of that angle?

The answer lies in using the system of coordinates we've developed. Even if you cannot draw a right triangle if the terminal arm is on an axis, you may still take a point on that line and calculate x, y, and r.

For instance, suppose we are interested in calculating the three basic trig functions for $\theta = 180°$. Let's look at the diagram below, and pick a point somewhere on the terminal arm, such as (–3, 0).



For any point we pick, the first coordinate must be a negative real number, since it lies on the negative x-axis, and the second coordinate must be zero since $y = 0$. For our particular point, r is +3, since the distance from the origin to the point we've chosen is 3.

Consequently,

$$\sin\theta = \frac{y}{r} = \frac{0}{3} = 0$$
$$\cos\theta = \frac{x}{r} = \frac{-3}{3} = -1$$
$$\tan\theta = \frac{y}{x} = \frac{0}{-3} = 0$$

It doesn't make sense to talk about the opposite, adjacent, and hypotenuse when there is no right triangle associated with the calculation, but the coordinates can still be placed in the appropriate places to calculate the trig functions, as we've done before.

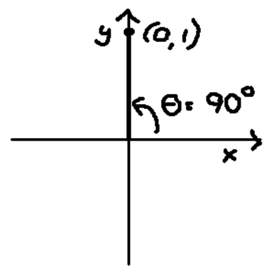
Even if we changed the coordinates of our point to something else, note that r will always just be the absolute value of the x-coordinate, since it's the distance, and even if we look at the point (–7, 0) or more generally $(x, 0)$ where x is negative, we'll still get the sine and tangent of the angle to equal zero and the cosine will equal negative one, because we'll be dividing the x value by the absolute value of x.

Example

Calculate the three basic trig functions of 90° using coordinates.

Answer

The terminal arm of $\theta = 90°$ lies on the $+y$-axis, as shown below.



We need to choose a point on the $+y$-axis. Let's choose (0, 1) for convenience. Then $r = 1$ (distance from origin to our point), and

$$\sin\theta = \frac{y}{r} = \frac{1}{1} = 1$$
$$\cos\theta = \frac{x}{r} = \frac{0}{1} = 0$$
$$\tan\theta = \frac{y}{x} = \frac{1}{0} = \text{undefined}$$

As you can see from the previous examples, the three basic trig functions of the quadrantal angles take on four possible values: +1, 0, –1, and "undefined". And. yes, it's quite possible for the tangent of an angle to be undefined. In fact, tan 90° and tan 270° and the tangent of any angle coterminal to these two will be undefined, since the x-coordinate must be zero.

Section 4.4: Trig Functions of Any Angle

Exercises

For the following angles in standard position, sketch the angle (including the swirly line to show direction of rotation).

1. 150°

2. 335°

3. 420°

4. –135°

5. –530°

6. 540°

For the following angles, list one positive and one negative coterminal angle. Answers may vary.

7. 25°

8. 210°

9. –300°

10. 1080°

11. –165°

12. –12°

Give the reference angle for the following angles.

13. 120°

14. –150°

15. 300°

16. –330°

17. 415°

18. 750°

State the signs (positive or negative) of the three basic trig functions of θ if θ is in

19. QI

20. QIV

21. QIII

22. QII

Calculate the following to four decimal places.

23. sin –63.2°

24. cos 154°

25. tan 90°

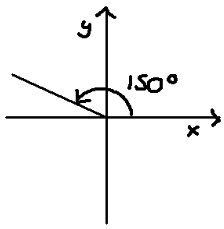
26. tan 1025°

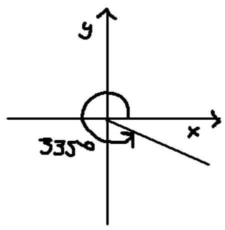
27. cos 180°

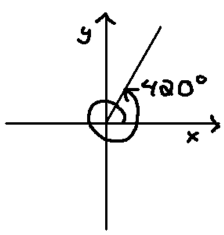
28. sin –52 515.82°

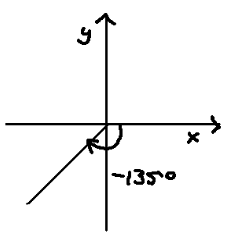
Section 4.4: Trig Functions of Any Angle

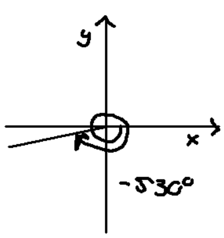
Answers

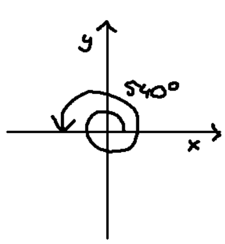
1. 

2. 

3. 

4. 

5. 

6. 

7. 385°, –335°

8. 570°, –150°

9. 60°, –660°

10. 360°, –360° (note: 0° is neither positive nor negative)

11. 195°, –525°

12. 348°, –372°

13. 60°

14. 30°

15. 60°

16. 30°

17. 55°

18. 30°

19. all three are +

20. cos θ is +, others negative

21. tan θ is +, others negative

22. sin θ is +, others negative

23. –0.8926

24. –0.8988

25. undefined

26. –1.4282

27. –1 (you could write it as –1.0000 if you like, but it's exact so I wrote it as an integer)

28. 0.6969