Section 5.1: Exponential Functions

We should begin by defining exponential functions. Recall that a function is a relation in which for every *x*-value, there is only one *y*-value. An exponential function, then, is a function of the form

$$y = a^x$$

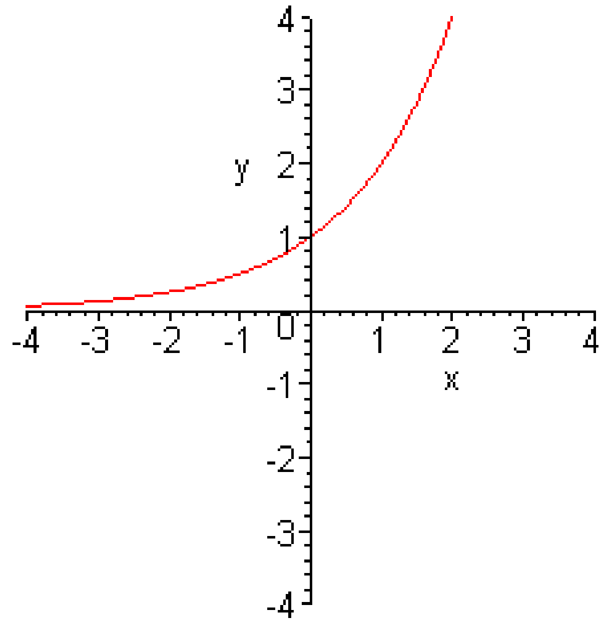
where the variable x is in the exponent. The constant a is then a real number with the restrictions that $a > 0$ and $a \neq 1$.

(We want $a > 0$ so that the graph of $y = a^x$ is a nice smooth curve – if we tried to graph $y = (-2)^x$, the graph would oscillate between positive and negative values since $(-2)^2$ equals +4 and $(-2)^3$ equals –8. The $a \neq 1$ restriction will be explained when we look at solving equations.)

Let's start by graphing $y = 2^x$. The brute force method is to draw up a table of values.

x	$y = 2^x$
-3	1/8
-2	¼
-1	½
0	1
1	2
2	4
3	8

By plotting these points and drawing a nice, smooth curve, we'll get a graph that looks like the following.



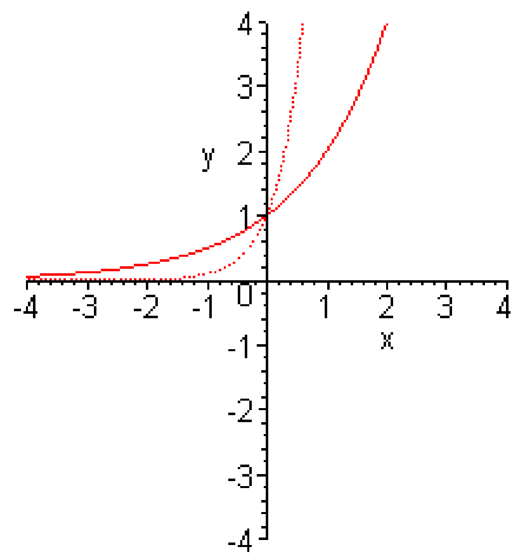
There are a few things to notice from this graph.

1. Note that the graph climbs very rapidly as *x* gets larger (this is what we really mean when we speak of "growing exponentially").

2. As *x* gets smaller (more negative), observe that *y* gets very small indeed but never touches the *x*-axis (we say that it approaches the axis asymptotically).

3. Notice that all values of *x* are allowed in the exponent, but the resulting *y* just be positive – never negative and never zero.

What if we were to graph $y = 10^x$ on the same graph? We'd get



where the solid line is $y = 2^x$ and the dotted line is $y = 10^x$.

The point (0,1) is common to both graphs: $y = a^x$ will contain the point (0,1) since $a^0 = 1$ for all a. The $y = 10^x$ line will simply rise more steeply as x gets larger, and will approach the x-axis more quickly as x gets more negative.

Solving Exponential Equations

We will start by solving a particular type of exponential equation in which you can match the bases. There is a nice property of exponential functions that will come in handy.

One-to-One Property of Exponential Functions

For $a > 0$ and $a \neq 1$,

if $a^m = a^n$, then $m = n$.

(Here's why we want $a \neq 1$: if $a = 1$, then $a^m = 1$ for all m and we can't find a unique answer for m.)

What this means is that if we can match the bases on each side of an equation, then we can match the exponents as well.

Example

Solve $3^{x+1} = 81$.

Answer: We start by trying to write each side using the same base. In this case, we can do this by substituting 3^4 for 81 on the right-hand side:

$3^{x+1} = 3^4$

If the bases are the same, then the exponents must also be the same. So,

$x + 1 = 4$

$x = 3$

So our solution set is {3}, and if we substitute it back into the original equation, we see that the left-hand side is $3^{3+1} = 3^4 = 81$. Check!

Example

Solve $2^x = \frac{1}{4}$.

Answer:

$$2^x = \frac{1}{4}$$
$$2^x = 2^{-2}$$
$$x = -2$$

And the solution set is {-2}.

Example

Solve $\left(\frac{1}{5}\right)^x = 125$.

Answer:

$$\left(\frac{1}{5}\right)^x = 125$$
$$\left(5^{-1}\right)^x = 5^3$$
$$5^{-x} = 5^3$$
$$-x = 3$$
$$x = -3$$

And the solution set is {-3}.

Example

Solve $16^x = 4$.

Answer:

$$16^x = 4$$
$$\left(4^2\right)^x = 4^1$$
$$4^{2x} = 4^1$$
$$2x = 1$$
$$x = 1/2$$

And the solution set is { ½ }.

Compound Interest

If a loan, an investment, or a mortgage is calculated with **compound interest**, that means that after a certain period (called the compounding period), interest is deposited into the account, and then interest is paid on that interest. Let's start out with principal P compounded annually with interest rate r. After one year, the amount A in the account will be the principal P plus interest $P \cdot r$, or $A = P(1+r)$. After the second year, then the new amount will be $P(1+r)^2$, and after t years, the total amount $A = P(1+r)^t$.

To generalize this to **any** compounding period, the interest rate for that compounding period will be r/n, where n is now the number of compounding periods per year. So our formula becomes

$$A = P\left(1+\frac{r}{n}\right)^{nt},$$

where A is the amount of investment after t years, P is the principal (the original amount of the investment), r is the annual interest rate, n is the number of compounding periods per year, and t is the total time of investment.

Example

If \$1000 is deposited into an account paying 10% per year compounded monthly, how much will be in the account after 10 years?

Answer:

$P = 1000$, $r = 0.1$, $n = 12$, and $t = 10$

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 1000\left(1+\frac{0.1}{12}\right)^{12\times 10} \\ &= 2707.04 \end{aligned}$$

So the account will contain the amount \$2707.04 after ten years.

Let's see how the amount after investment varies as a function of the compounding period. Consider the following table.

	A	B	C	D
1	*compounding period*		$1000\left(1+\frac{0.1}{x}\right)^{(10\,x)}$	
2	*yearly*	1	2593.7424601000	
3	*monthly*	12	2707.0414908622531148	
4	*weekly*	52	2715.6726950308601142	
5	*daily*	365	2717.9095545777538264	
6	*hourly*	8760	2718.2663133141708018	
7	*every minute*	525600	2718.2815698708146523	
8	*every second*	31536000	2718.2818241694182054	
9				
10	*continuously*		2718.2818284590452354	

You can see that when the compounding period gets sufficiently small, the amount gets closer and closer to a particular value. This value is the continuous limit, and can be given by the equation

$$A = Pe^{rt}$$

where A is the amount of the investment, P is the principal, r is the interest rate, and t is the time of the investment. The number e is a constant, equal to 2.71828182845904… (which is the continuous limit above divided by our initial investment of $1000). This number e is irrational, so when you represent it by a decimal, the decimal equivalent does not repeat and does not terminate (exactly like the decimal equivalent of π).

The equation for the continuous limit does not just apply to investments – it is the equation for all types of exponential growth. If you have a population of mosquitoes or fluffy bunnies or bacteria and that population has enough resources to grow unchecked, then it will grow according to that equation, in other words **exponentially**.

Section 5.1: Exponential Functions

Exercises

Sketch the following graphs. (I recommend calculating a table of values as your first step.)

1. $y = 2^x$

2. $y = 3^x$

3. $y = \left(\frac{1}{2}\right)^x$

4. $y = 2^{-x}$

5. $y = 4^x$ and $y = \left(\frac{1}{4}\right)^x$ on the same graph

Solve the following exponential equations.

6. $3^{x+2} = 9$

7. $6^x = \frac{1}{36}$

8. $10^{-x} = 0.01$

9. $4^{5-x} = 64$

10. $8^x = \frac{1}{2}$

11. $5^{2x} = 125$

12. $2^{5+x} = 256$

13. $64^{5+x} = 4$

14. $100^{5-x} = 1000^2$

15. $49^{2m} = 7^{m+1}$

16. $2^{b+1} = 8^{1-b}$

17. $4^x = \sqrt{2}$

18. $\left(\sqrt{2}\right)^y = 4$

19. $\left(\sqrt{2}\right)^k = \frac{1}{2}$

20. $0.1^x = 100$

21. $0.5^{0.5x} = 16$

22. $5^{2x} = 5^{3x}$

Solve the following word problems.

23. Nicole has invested $5000 into an account paying 5% per year, compounded semi-annually. How much money will be in her account after five years?

24. Peter has a high-tech savings account that compounds continuously. How much money will he have for an initial investment of $800 at 3% per year after six years?

25. Darcy has borrowed $2000 from his bank at 8% per year compounded daily. How much will he owe the bank after one year if he pays everything off at once?

26. A mutual fund at Fred's Bank returns 12% annually. How much will an initial investment of $10,000 be worth after ten years?

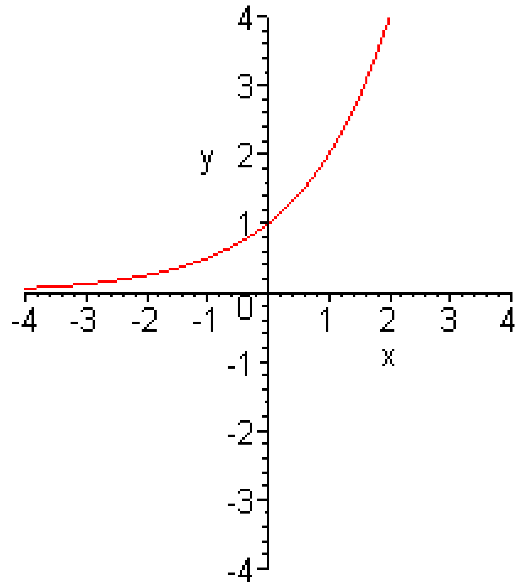
27. David has invested $1500 into an account paying 2% per year. How much money will he have after 3 years if the interest is compounded

 a) yearly?
 b) weekly?
 c) daily?
 d) continuously?

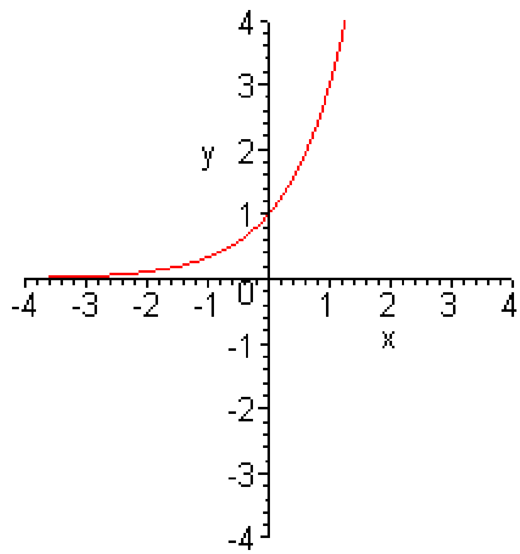
Section 5.1: Exponential Functions

Solutions to all questions

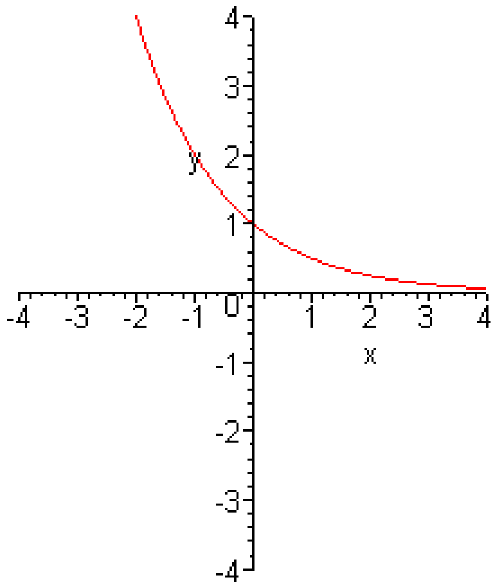
1. $y = 2^x$



2. $y = 3^x$

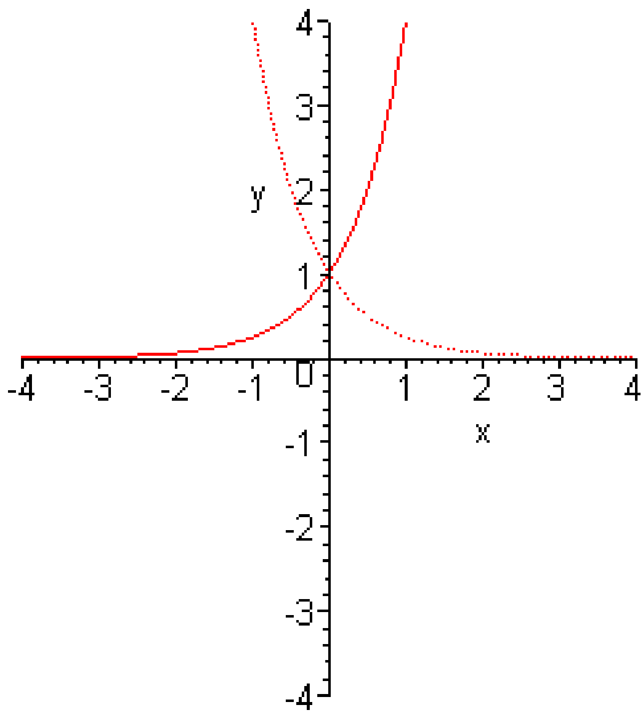


3. $y = \left(\frac{1}{2}\right)^x$



4. $y = 2^{-x}$ – same as graph for question 3, since $y = 2^{-x} = \left(2^{-1}\right)^x = \left(\frac{1}{2}\right)^x$

5. $y = 4^x$ is the solid line and $y = \left(\frac{1}{4}\right)^x$ is the dotted line



6.
$$\begin{aligned} 3^{x+2} &= 9 \\ 3^{x+2} &= 3^2 \\ x+2 &= 2 \\ x &= 0 \end{aligned}$$

7.
$$\begin{aligned} 6^x &= \frac{1}{36} \\ 6^x &= 6^{-2} \\ x &= -2 \end{aligned}$$

8.
$$\begin{aligned} 10^{-x} &= 0.01 \\ 10^{-x} &= 10^{-2} \\ -x &= -2 \\ x &= 2 \end{aligned}$$

9.
$$\begin{aligned} 4^{5-x} &= 64 \\ 4^{5-x} &= 4^3 \\ 5-x &= 3 \\ 2 &= x \\ x &= 2 \end{aligned}$$

10.
$$\begin{aligned} 8^x &= \frac{1}{2} \\ \left(2^3\right)^x &= 2^{-1} \\ 2^{3x} &= 2^{-1} \\ 3x &= -1 \\ x &= -\frac{1}{3} \end{aligned}$$

11.
$$\begin{aligned} 5^{2x} &= 125 \\ 5^{2x} &= 5^3 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

12.
$$\begin{aligned}&2^{5+x}=256\\&2^{5+x}=2^{8}\\&5+x=8\\&x=3\end{aligned}$$

13.
$$\begin{aligned}&64^{5+x}=4\\&\left(4^{3}\right)^{5+x}=4^{1}\\&4^{3(5+x)}=4^{1}\\&3(5+x)=1\\&15+3x=1\\&3x=-14\\&x=-\frac{14}{3}\end{aligned}$$

14.
$$\begin{aligned}&100^{5-x}=1000^{2}\\&\left(10^{2}\right)^{5-x}=\left(10^{3}\right)^{2}\\&10^{2(5-x)}=10^{6}\\&2(5-x)=6\\&10-2x=6\\&4=2x\\&x=2\end{aligned}$$

15.
$$\begin{aligned}&49^{2m}=7^{m+1}\\&7^{4m}=7^{m+1}\\&4m=m+1\\&3m=1\\&m=\frac{1}{3}\end{aligned}$$

16.
$$\begin{aligned}&2^{b+1}=8^{1-b}\\&2^{b+1}=2^{3(1-b)}\\&b+1=3(1-b)\\&b+1=3-3b\\&4b=2\\&b=\frac{1}{2}\end{aligned}$$

17. $$\begin{aligned} &4^x = \sqrt{2} \\ &2^{2x} = 2^{1/2} \\ &2x = 1/2 \\ &x = 1/4 \end{aligned}$$

18. $$\begin{aligned} &\left(\sqrt{2}\right)^y = 4 \\ &2^{y/2} = 2^2 \\ &y/2 = 2 \\ &y = 4 \end{aligned}$$

19. $$\begin{aligned} &\left(\sqrt{2}\right)^k = \frac{1}{2} \\ &2^{k/2} = 2^{-1} \\ &k/2 = -1 \\ &k = -2 \end{aligned}$$

20. $$\begin{aligned} &0.1^x = 100 \\ &10^{-x} = 10^2 \\ &-x = 2 \\ &x = -2 \end{aligned}$$

21. $$\begin{aligned} &0.5^{0.5x} = 16 \\ &(1/2)^{x/2} = 2^4 \\ &2^{-x/2} = 2^4 \\ &-x/2 = 4 \\ &x = -8 \end{aligned}$$

22. $$\begin{aligned} &5^{2x} = 5^{3x} \\ &2x = 3x \\ &0 = 3x - 2x \\ &0 = x \\ &x = 0 \end{aligned}$$

23. $$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 5000\left(1+\frac{0.05}{2}\right)^{2\times 5} \\ &= 6400.42 \end{aligned}$$ So Nicole will have \$6400.42 in her account after 5 years.

24. $$\begin{aligned} A &= Pe^{rt} \\ &= 800e^{0.03\times 6} \\ &= 957.77 \end{aligned}$$ Peter will have \$957.77 after six years.

25. $$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 2000\left(1+\frac{0.08}{365}\right)^{365\times 1} \\ &= 2166.56 \end{aligned}$$ Darcy owes the bank \$2166.56 after one year. (Note: you will still get the same answer if you use 365.25 days in a year. And now that I've reread the question, wouldn't he owe the bank zero dollars if he's already paid the loan off?)

26. $$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 10000\left(1+\frac{0.12}{1}\right)^{10} \\ &= 31058.48 \end{aligned}$$ The investment will be worth \$31 058.48 after ten years.

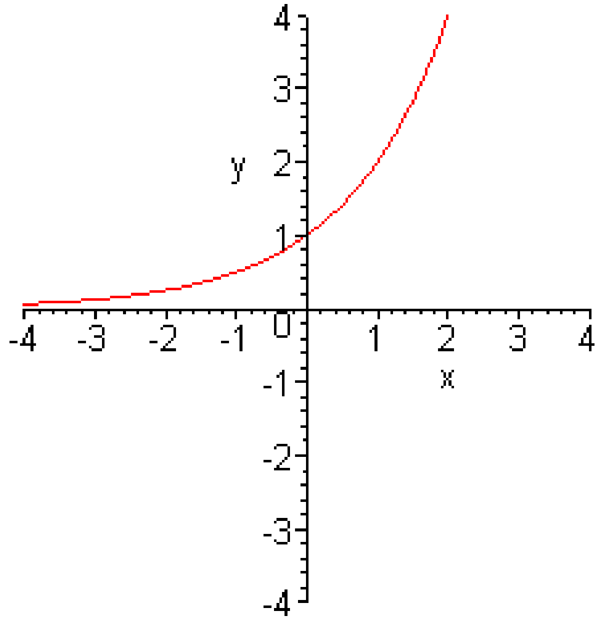
27. For parts a, b, and c, the equation will be $$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 1500\left(1+\frac{0.02}{n}\right)^{3n} \end{aligned}$$.

a) $n = 1$, so $A = 1591.81$ David will have \$1591.81 after three years.
b) $n = 52$, so $A = 1592.74$ David will have \$1592.74 after three years.
c) $n = 365$, so $A = 1592.75$ David will have \$1592.75 after three years.

d) Compounding continuously requires the equation $$\begin{aligned} A &= Pe^{rt} \\ &= 1500e^{0.02\times 3} \\ &= 1592.75 \end{aligned}$$, so David will have \$1592.75 after three years (no significant difference between this and daily compounding).

Section 5.2: Logarithmic Functions

Let's look at the $y = 2^x$ graph from the last section:



You'll notice that we plotted points for **integer** values of *x*, which led to either integer values for *y* or some familiar fractions like ½, ¼ and so on. However, when we drew the curved line through the points, we **implied** that *x* could take on any real values, including fractions, decimals or irrational numbers.

For example, in the graph above when $x = 0.8$, what's the corresponding value of *y*? We can draw a line upwards from the *x*-axis at $x = 0.8$ to find that it intersects the curve when *y* is approximately equal to 1.7. This means that $2^{0.8} \approx 1.7$ (the squiggly equals-sign means "approximately equal to"). We'll find out later how to calculate this exactly.

Also, we can ask, "For what value of *x* does $2^x = 3$?" From the graph, we see that when $y = 3$, the *x*-value is about 1.6 or so. The question, then, is how can we perform these calculations exactly instead of estimating from a graph? In the last section, you'll recall that we did many examples of solving similar equations like $2^x = 4$ exactly. The technique we used was to rewrite the right-hand side so that the bases matched. However, we can't use that technique here, because we do not know how to write the number 3 as 2 raised to some integer or fraction.

Instead, we have to use a new function: the **logarithm**.

Inverses

Before we examine the logarithm, we should talk first about functions and their inverses. For example, let's take the function $y = 2x + 4$. If for some strange reason, we wanted to solve for *x* instead, we'd perform the following procedure:

$$y = 2x + 4$$
$$y - 4 = 2x$$
$$\frac{y-4}{2} = \frac{2x}{2}$$
$$\frac{1}{2}y - 2 = x$$
$$x = \frac{1}{2}y - 2$$

and then $x = \frac{1}{2}y - 2$ is the inverse to our original function. Let's find some others: if $y = x^3$, then $x = \sqrt[3]{y}$ is the inverse. If $y = \sin x$, then $x = \sin^{-1} y$. The logarithm function is then the inverse to the exponential function: it's what we get if we solve the equation $y = a^x$ for *x*.

Logarithms

Let's present the exponential equation along with its inverse:

$$y = a^x$$
$$\log_a y = x$$

These two equations are inverses of each other. To get to the second (scary-looking) one from the first, move the *x* in the exponent down so it's by itself on the right-hand side. The base, *a*, then moves to the other side and becomes the base of the logarithm, and the *y* is then the argument of the logarithm (what the logarithm is operating on). This probably doesn't seem particularly enlightening, so let's move to a numerical example.

Example

Solve the following equation for *x*: $10^x = 3$. Give both an exact answer and a decimal approximation to two decimal places.

Answer:

We want to move the base, 10, to the other side as the base of the logarithm. We then get:

$$10^x = 3$$
$$x = \log_{10} 3$$

This is the exact answer, $x = \log_{10} 3$. To calculate it exactly, you'll want a scientific calculator with a log button. If you enter "log 3" (on some calculators, you'll enter 3 and then hit log), then you'll get 0.477121 with a few more decimals. Rounding to two decimal places gives x = 0.48. The log button on scientific calculators means "$\log_{10}$". We'll find out how to calculate logs with different bases later.

How can we check? Our original equation was $10^x = 3$. By plugging $10^{0.477121}$ into the calculator, we should get 3. (If you use the rounded-off answer of 0.48, you'll get a number that's a little more than 3, 3.01995.) So we can see that we have arrived at the decimal approximation to our original equation.

Note, also, that we should expect an answer somewhere between zero and one. Why? Well, 10^0 is equal to 1 and 10^1 = 10, so finding the exponent that will give an answer of 3 must lie between 0 and 1.

Example

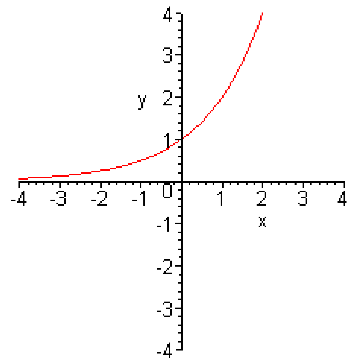
Solve the following equation for x: $10^x = 4200$. Give both an exact answer and a decimal approximation to two decimal places.

Answer:

$$10^x = 4200$$
$$x = \log_{10} 4200$$
$$x \approx 3.62$$

Again, we should expect an answer between 3 and 4, because 10^3 = 1000 and 10^4 = 10,000.

Now, logs can have any base that obeys the rules we studied for bases of exponents in the last section, namely that our base a must be greater than zero and not equal to one. But there are some properties that we should be aware of. Let's look once more at the $y = 2^x$ graph.



Notice that we can plug any real number into *x*, the exponent of $y = 2^x$, but the resulting *y* is always positive. For the inverse function, we are putting in the *y*-value and finding the resulting exponent. So the logarithm function $\log_a y = x$ can only having positive values for *y* applied to it, but the resulting *x* can be any real number.

Example

Calculate $\log_{10} 100$, $\log_{10} 0.1$, $\log_{10} 0$, and $\log_{10}(-10)$ using your calculator.

Answer:

Remembering that the "log" button on your calculator finds logs of base 10, you can just type in the above numbers. Your calculator should find that $\log_{10} 100 = 2$. This should make sense, because if you set $\log_{10} 100$ to a variable and then change the equation into exponential form, you get

$$x = \log_{10} 100$$
$$10^x = 100$$

But 100 is just 10 squared, so *x* must equal 2, using our matching-bases technique from the previous section.

Using the calculator for $\log_{10} 0.1$, you should get -1 (which should make sense because 10^{-1} = 0.1. However, trying to calculate $\log_{10} 0$ and $\log_{10}(-10)$ should give you errors on your calculator. That is because the *y*-value of the logarithm function $\log_a y$ must always be positive.

Natural Logs and Common Logs

Although logarithms can have any base provided that it's positive and not equal to one, the two most common bases are 10 and *e*. Because they are used so frequently, there are some conventions we should examine. You've already notice one: even though the button on your calculator calculates logs with base of 10, it doesn't have the number 10 explicitly written. That's because the convention is that when the base is missing, the base meant is 10, and the logarithm with base 10 is called the **common log**.

(It's like the convention that the variable x is actually x^1, but we commonly omit the exponent if it's equal to one. Also, for radicals, if the index n is missing, then it's assumed to be equal to 2: $\sqrt{x} = \sqrt[2]{x}$. If we are taking some other root, like the cube root, we have to remember to write the index for the radical sign.)

To write logarithms with base of e, we replace the "$\log_e$" by just "ln", since the log with base e is called the **natural log**. So $\log_e x = \ln x$, and your calculator should also have a button with "ln" on it.

Example

Solve the following equation for x: $e^x = 24$. Give both an exact answer and a decimal approximation to two decimal places.

Answer:

$$\begin{aligned} e^x &= 241 \\ x &= \ln 241 \\ x &\approx 5.48 \end{aligned}$$

Example

Solve the following equation for x: $e^{2x+7} = 24$. Give both an exact answer and a decimal approximation to two decimal places.

Answer:

$$\begin{aligned} e^{2x+7} &= 24 \\ 2x+7 &= \ln 24 \\ 2x &= \ln 24 - 7 \\ x &= \frac{\ln 24 - 7}{2} \\ x &\approx -1.91 \end{aligned}$$

So $x = \dfrac{\ln 24 - 7}{2}$ is the exact answer and –1.91 is the approximate answer.

Exact Calculations

Let's do a little more work on understanding what the various parts of the logarithm mean. For example, what does $\log_2 8$ mean?

$$x = \log_2 8$$
$$2^x = 8$$
$$2^x = 2^3$$
$$x = 3$$

So $\log_2 8$ means "what do you have to raise 2 to the power of to get 8?"

Example

Simplify the following:

a) $\log_3 27$

b) $\log_{10} \frac{1}{100}$

c) $\log_6 1$

d) $\log_5 0$

Answer:

a) $3^3 = 27$, so we must raise 3 to the power of **3** to get 27, so the logarithm equals 3.

b) $10^{-2} = \frac{1}{100}$, so $\log_{10} \frac{1}{100} = -2$.

c) $6^0 = 1$, so $\log_6 1 = 0$.

d) You can't put non-positive numbers into logarithms, so $\log_5 0$ is undefined.

Example

Simplify the following:

a) $\log_a a^4$

b) $\log_a a$

c) $\log_a 1$

Answer:

a) To get a^4, we must raise *a* to the fourth power, so $\log_a a^4$ must be 4.

b) $a^1 = a$, so $\log_a a = 1$.

c) $a^0 = 1$, so $\log_a 1 = 0$.

Logarithmic vs. Exponential Forms

As we have seen, every exponential equation has a corresponding logarithmic equation. We should practice transforming between the two forms.

Example

Write the corresponding exponential/logarithmic forms for the following equations. Don't simplify or solve!

a) $a = \log_b c$

b) $q = m^n$

Answer

a) This equation is in logarithmic form. To get it to exponential form, move the base to the other side of the equation, and the variable that's there already gets bumped up into the exponent: $b^a = c$.

b) This equation is in exponential form. To convert it to logs, the base *m* moves to the other side and becomes the base of the logarithm, $\log_m q = n$.

Section 5.2: Logarithmic Functions

Exercises

Solve for *x*. Give both an exact answer (with "$\log_{10}$" or just "log" in your answer) and a decimal approximation.

1. $10^x = 0.75$
2. $10^x = 12\,345$
3. $10^{2x} = 37$
4. $10^{x+5} = 23.7$
5. $10^{1-x} = 0.002$
6. $10^{2x+3} = 8$

Solve for *x*. Give both an exact answer (containing "ln") and a decimal approximation.

7. $e^{x+2} = 9$
8. $e^{-2x} = 18$
9. $e^{3x-1} = 0.7$
10. $e^{x-4} = 1$
11. $e^{2-5x} = 157$
12. $e^{x+1} = \pi$

Simplify.

13. $\log_2 4$
14. $\log_{10} 0.0001$
15. $\log_7 1$
16. $\log_2 \frac{1}{8}$

17. $\log_2(-4)$

18. $\log_3 81$

19. $\log_{20} 20$

20. $\log_{1000} 10$

Write the corresponding exponential/logarithmic equation. Don't simplify!

21. $y = e^x$

22. $y = \log_2 x$

23. $f = h^2$

24. $m = n^p$

25. $y = \log_x 2$

26. $10 = k^m$

27. $x = \log_2 y$

28. $8 = \log_6 c$

Section 5.2: Logarithmic Functions

Answers

1. $x = \log(0.75) = -0.125$ (Here I've chosen to round to 3 decimals for my answers, but as I didn't specify in my question how many I wanted, anything reasonable will do.)

2. $x = \log(12\,345) = 4.091$

3. $x = \frac{1}{2}\log(37) = 0.784$

4. $x = \log(23.7) - 5 = -3.625$

5. $x = 1 - \log(0.002) = 3.699$

6. $x = \dfrac{\log(8) - 3}{2} = -1.048$

7. $x = \ln 9 - 2 = 0.197$

8. $x = -\frac{1}{2}\ln(18) = -1.445$

9. $x = \dfrac{\ln 0.7 + 1}{3} = 0.214$

10. $x = 4$ (this answer is exact, so doesn't need a decimal approximation)

11. $x = \dfrac{2 - \ln(157)}{5} = -0.611$

12. $x = \ln \pi - 1 = 0.145$

13. 2

14. –4

15. 0

16. –3

17. undefined

18. 4

19. 1

20. 1/3

21. $\ln y = x$

22. $2^y = x$

23. $\log_h f = 2$

24. $\log_n m = p$

25. $x^y = 2$

26. $\log_k 10 = m$

27. $2^x = y$

28. $6^8 = c$

Section 5.3: Properties of Logarithms

Let's examine some properties of logarithms that will allow us to solve equations containing logs more easily.

The Product Rule

Let's start with a numerical example to develop the ideas behind the product rule of logarithms. First, let's calculate the following logarithms (using a calculator!), remembering that "log" means "$\log_{10}$".

$$\begin{aligned}&\log 5.7\\&\log 57\\&\log 570\\&\log 5700\end{aligned}$$

We find that

$$\begin{aligned}\log 5.7 &= 0.755875\\ \log 57 &= 1.755875\\ \log 570 &= 2.755875\\ \log 5700 &= 3.755875\end{aligned}$$

and when we look at the numbers, we can see a pattern developing. We can rewrite the numbers on the right-hand side to get

$$\begin{aligned}\log 57 &= 1 + 0.755875\\ \log 570 &= 2 + 0.755875\\ \log 5700 &= 3 + 0.755875\end{aligned}$$

which may not seem particularly enlightening until we remember that $1 = \log 10$ and notice also that 0.755875 is just log 57. We then get

$$\begin{aligned}\log 57 &= \log 10 + \log 5.7\\ \log 570 &= \log 100 + \log 5.7\\ \log 5700 &= \log 1000 + \log 5.7\end{aligned}$$

and finally that

$$\begin{aligned}\log(10 \times 5.7) &= \log 10 + \log 5.7\\ \log(100 \times 5.7) &= \log 100 + \log 5.7\\ \log(1000 \times 5.7) &= \log 1000 + \log 5.7\end{aligned}$$

In other words, if you are taking the logarithm of a product, it's equal to the sum of the logs of the individual terms of the product. Writing that in symbols, we get

$$\log_a(MN) = \log_a M + \log_a N \ .$$

Note, however, that if you wish to combine two logarithms into a single one using this property, the **bases must be the same**!

Example

Express as a single logarithm:

a) $\log x + \log y$

b) $\ln 2 + \ln 3$

c) $\log_b x^2 + \log_b x^3$

d) $\log 24x^2 + \log \frac{x}{8}$

Answers:

a) $\log xy$

b) $\ln 6$

c) $\log_b x^5$

d) $\log 3x^3$

Example

Use the product rule to write an equivalent expression for the following:

a) $\log_b 2x$

b) $\ln mn$

c) $\log 6p$

d) $\log_b 3pq$

Answers:

a) $\log_b 2 + \log_b x$

b) $\ln m + \ln n$

c) $\log 6 + \log p$

d) $\log_b 3 + \log_b p + \log_b q$

The Power Rule

In math, we like to take any new problem and if possible, rewrite it so that it looks like a problem we know how to solve. Using this idea, we can rewrite

$$\ln x^3 = \ln(x \cdot x \cdot x)$$

Then we use the product rule to rewrite the right-hand side as

$$\ln x^3 = \ln x + \ln x + \ln x$$

and then we can use algebra to collect the like terms on the right-hand side to get

$$\ln x^3 = 3\ln x \text{ .}$$

Generalizing, if we have the logarithm of a number raised to a power, we can apply the power rule:

$$\log_a M^N = N \log_a M \text{ .}$$

Example

Use the power rule to write an equivalent expression for the following:

a) $\log y^{10}$

b) $\ln 2^x$

c) $\log_3 5^7$

d) $\log \sqrt{y}$

e) $\log_x p^q$

f) $\ln m^{-1}$

Answers:

a) $10\log y$

b) $x\ln 2$

c) $7\log_3 5$

d) $\frac{1}{2}\log y$

e) $q\log_x p$

f) $-\ln m$

The Quotient Rule

Once again, we will try to use our previous ideas to develop a property of logarithms. Consider the logarithm of a quotient. We can try to rewrite the quotient to be a product instead, since we know the product rule.

$$\log_a\left(\frac{M}{N}\right)=\log_a\left(M\cdot N^{-1}\right)$$

We then use the product rule to expand the right-hand side.

$$\log_a\left(\frac{M}{N}\right)=\log_a M+\log_a\left(N^{-1}\right)$$

But then the last term can be simplified using the power rule to give the quotient rule:

$$\log_a\left(\frac{M}{N}\right)=\log_a M-\log_a N\ .$$

Example

Use the quotient rule to write an equivalent expression for the following:

a) $\log\frac{x}{4}$

b) $\ln\frac{a}{b}$

c) $\ln\frac{x}{yz}$

d) $\log_b \frac{2m}{n}$

Answers:

a) $\log x - \log 4$

b) $\ln a - \ln b$

c) $\ln x - \ln y - \ln z$

d) $\log_b 2 + \log_b m - \log_b n$

Simplification

Remember that, according to our original definition of logarithm, that $\log_a b$ means "what do I have to raise *a* to the power of to get *b*?" Therefore,

$$\log_a a^x = x$$

Taking this idea a step further, then for any base *a*, $a = a^1$ so that $\log_a a = \log_a a^1 = 1$, and generally

$$\log_a a = 1.$$

Similarly, $\log_a 1 = \log_a a^0 = 0$, so

$$\log_a 1 = 0$$

for any base *a*.

Example

Simplify:

a) $\log_x x^4$

b) $\log_a \sqrt[3]{a}$

c) $\ln e^x$

d) $\log 10^3$

e) $\ln \sqrt[5]{e}$

f) $\log_b \frac{1}{b}$

Answers:

a) 4

b) 1/3

c) x

d) 3

e) 1/5

f) –1

Example

Write each expression as a single logarithm and simplify:

a) $\log_b x^2 + \log_b x^{-2}$

b) $5\log x - 2\log y$

c) $\ln 8 - \ln 2$

d) $\log 2 + \log 5$

e) $\frac{1}{2}\log 400 - \log 2$

Answers:

a) 0

b) $\frac{x^5}{y^2}$

c) $\ln 4$

d) 1

e) 1

Example

Write each expression in terms of ln 3 and/or ln x:

a) $\ln\frac{1}{3}$

b) $\ln\sqrt{3}$

c) $\ln 27x$

d) $\ln(3x)^5$

Answers:

a) –ln 3

b) ½ ln 3

c) 3 ln 3 + ln x

d) 5 ln 3 + 5 ln x or 5(ln 3 + ln x)

Example

Given that $\log_a x = 2$ and $\log_a y = 3$, evaluate:

a) $\log_a xy$

b) $\log_a \frac{y}{x}$

c) $\frac{\log_a y}{\log_a x}$

d) $\log_a \sqrt{x}$

e) $\log_a x^3$

Answers:

a) 5

b) 1

c) 3/2

d) 1

e) 6

The Base-Change Formula

Up until now, we've only been able to calculate decimal equivalents for logarithms with base 10 or *e*, since those are the only two bases available on our calculators. Now let's learn a method to calculate the decimal approximation for logarithms of any base. Suppose we had the equation

$$a^x = M$$

and tried to solve it in two different ways. Our first way might be to rewrite the equation into the equivalent logarithmic form by bringing the base *a* down to get

$$x = \log_a M \ .$$

But we could also try taking the logarithm base 10 of both sides of the original equation, like so:

$$a^x = M$$
$$\log a^x = \log M$$

Then apply the power rule to get

$$x \log a = \log M$$

and solve for *x*:

$$x = \frac{\log M}{\log a}$$

As these two methods are equivalent, the two answers we get are equivalent and

$$\log_a M = \frac{\log M}{\log a} .$$

More generally, for any positive number M and any bases *a* and *b*, the base-change formula is:

$$\log_a M = \frac{\log_b M}{\log_b a} \quad \left(= \frac{\log M}{\log a} = \frac{\ln M}{\ln a} \right)$$

Example

Use the base-change formula to calculate each logarithm below to four decimal places:

a) $\log_3 5$

b) $\log_{12} 0.3$

c) $\log_{0.2} 9000$

d) $\log_{0.1} 0.3$

Answers:

a) $\log_3 5 = \dfrac{\log 5}{\log 3} = 1.4650$

b) $\log_{12} 0.3 = \dfrac{\log 0.3}{\log 12} = -0.4845$

c) $\log_{0.2} 9000 = \dfrac{\log 9000}{\log 0.2} = -5.6572$

d) $\log_{0.1} 0.3 = \dfrac{\log 0.3}{\log 0.1} = 0.5229$

Note that you can check your answers: if you take the last example and calculate $0.1^{0.5229}$, you get 0.299985, which is almost equal to 0.3. The reason it's not **exactly** equal is because 0.5229 has been rounded off to four decimal places. If you check by using the full number in your calculator, 0.52287874528033, you should get a better approximation, and your calculator will round that answer to exactly 0.3. Check!

Section 5.3: Properties of Logarithms

Exercises

State whether the following equations are T (true) or F (false) for all possible values of the variables. You'll note that some of these are indeed the properties of logs. However, some only **look** like the properties of logs, but unfortunately are incorrect.

1. $\log(M+N) = \log M + \log N$

2. $\log(MN) = (\log M)(\log N)$

3. $\log(MN) = \log M + \log N$

4. $n \log x = \log(nx)$

5. $\dfrac{\log M}{\log N} = \log\left(\dfrac{M}{N}\right)$

6. $\dfrac{\log M}{\log N} = \dfrac{M}{N}$ (logs cancel)

7. $\dfrac{\log M}{\log N} = \log_N M$

8. $n \log x = \log(x^n)$

Write each expression as a single logarithm and simplify.

9. $\log 2000 - \log 2$

10. $\log 2 + \log 5$

11. $\log_2 4x - \log_2 x$

12. $\log_3 5 + \log_3 2$

13. $\log_5 x - \log_5 y + \log_5 z$

14. $\log_a a^3 - 2\log_a a$

Write each expression in terms of log 2 and/or log *x*.

15. $\log 2x$

16. $\log\left(\frac{x}{4}\right)$

17. $\log\left(8x^3\right)$

18. $\log\sqrt{x}$

19. $\log\left(\frac{1}{x}\right)$

20. $\log\left(2x\right)^7$

Rewrite each expression as a single logarithm and simplify.

21. $2\log x - 3\log y$

22. $\frac{1}{3}\log x + 5\log 2$

23. $2\log 3 - 3\log y$

24. $\log 5 + 3\log 2 - \log 4$

25. $\frac{1}{2}\log 4 + \frac{1}{3}\log 27$

26. $2\log x + 3\log x^2$

Use the base-change formula to find each logarithm to four decimal places

27. $\log_7 10$

28. $\log_{0.2} 15$

29. $\log_{1.05} 2$

30. $\log_2 1.05$

31. $\log_{500} 1000$

32. $\log_{0.001} 10$

Section 5.3: Properties of Logarithms

Solutions

1. $\log(M+N)=\log M+\log N$ – False

2. $\log(MN)=(\log M)(\log N)$ – False

3. $\log(MN)=\log M+\log N$ – True (this is the product rule)

4. $n\log x=\log(nx)$ – False

5. $\dfrac{\log M}{\log N}=\log\left(\dfrac{M}{N}\right)$ – False

6. $\dfrac{\log M}{\log N}=\dfrac{M}{N}$ (logs cancel) – False

7. $\dfrac{\log M}{\log N}=\log_N M$ – True (this is the base-change formula)

8. $n\log x=\log(x^n)$ – True (this is the power rule)

9. $\log 2000-\log 2=\log\dfrac{2000}{2}=\log 1000=3$

10. $\log 2+\log 5=\log 10=1$

11. $\log_2 4x-\log_2 x=\log_2\dfrac{4x}{x}=\log_2 4=2$

12. $\log_3 5+\log_3 2=\log_3 10$ (note: doesn't simplify further)

13. $\log_5 x-\log_5 y+\log_5 z=\log_5\left(\dfrac{xz}{y}\right)$

14. $\log_a a^3-2\log_a a=3\log_a a-2\log_a a=\log_a a=1$

15. $\log 2x=\log 2+\log x$

16. $\log\left(\dfrac{x}{4}\right)=\log x-\log 4=\log x-\log 2^2=\log x-2\log 2$

17. $\log\left(8x^3\right) = \log 8 + \log x^3 = \log 2^3 + 3\log x = 3\log 2 + 3\log x$

18. $\log\sqrt{x} = \log x^{1/2} = \frac{1}{2}\log x$

19. $\log\left(\frac{1}{x}\right) = \log x^{-1} = -\log x$

20. $\log\left(2x\right)^7 = 7\log(2x) = 7(\log 2 + \log x) = 7\log 2 + 7\log x$

21. $2\log x - 3\log y = \log x^2 - \log y^3 = \log\left(\frac{x^2}{y^3}\right)$

22. $\frac{1}{3}\log x + 5\log 2 = \log x^{1/3} + \log 2^5 = \log\left(32x^{1/3}\right)$

23. $2\log 3 - 3\log y = \log 3^2 - \log y^3 = \log\left(\frac{9}{y^3}\right)$

24. $\log 5 + 3\log 2 - \log 4 = \log 5 + \log 2^3 - \log 4 = \log\frac{5 \bullet 8}{4} = \log 10 = 1$

25. $\frac{1}{2}\log 4 + \frac{1}{3}\log 27 = \log 4^{1/2} + \log 27^{1/3} = \log 2 + \log 3 = \log 6$

26. $2\log x + 3\log x^2 = 2\log x + 6\log x = 8\log x$ (or $\log x^8$)

27. $\log_7 10 = \frac{\ln 10}{\ln 7} = 1.183$ (note that you could also do a ratio of log rather than ln)

28. $\log_{0.2} 15 = \frac{\ln 15}{\ln 0.2} = -1.683$

29. $\log_{1.05} 2 = \frac{\ln 2}{\ln 1.05} = 14.207$

30. $\log_2 1.05 = \frac{\ln 1.05}{\ln 2} = 0.0703$

31. $\log_{500} 1000 = \frac{\ln 1000}{\ln 500} = 1.112$

32. $\log_{0.001} 10 = \frac{\ln 10}{\ln 0.001} = -0.333$

Section 5.4: Solving Equations

In this section, we'll look at solving both exponential and logarithmic equations.

Solving Exponential Equations

Recall that an exponential function is a function in which the variable is in the exponent. Similarly, exponential equations have the variable in the exponent as well. Examples are

$$2^{x+1} = 64 \text{ and } 3^x = 20 \text{ .}$$

Now, we can solve the first one using the one-to-one property of exponential functions, which says that if the bases match, then so do the exponents. Rewriting, we find that

$$\begin{aligned} &2^{x+1} = 64 \\ &2^{x+1} = 2^6 \\ &x+1 = 6 \\ &x = 5 \end{aligned}$$

Now, let's look at the second one. We cannot use the previous procedure here because we can't match the bases (that would require writing 20 as some power of 3, which we don't know how to do). Therefore, we have to use logarithms.

$$\begin{aligned} &3^x = 20 \\ &x = \log_3 20 \end{aligned}$$

Now, the solution $\log_3 20$ is an exact answer. We could use the base-change formula to get it into a form that we can approximate on our calculator, like so:

$$x = \frac{\log 20}{\log 3} \approx 2.72683$$

Notice, though, that we could have come up with the same solution using a different method. Instead of changing the exponential equation to a logarithmic equation of the same base, what would happen if we took the logarithm, base 10, of both sides:

$$\begin{aligned} &3^x = 20 \\ &\log 3^x = \log 20 \end{aligned}$$

Using the power rule on the left-hand side, we get

$$x \log 3 = \log 20$$
$$x = \frac{\log 20}{\log 3}$$

which you'll notice is the same solution as the one we got after using the base-change formula. Then if we were to plug this into our calculators, we should get the same decimal approximation as before.

But what if we have an expression in the exponent instead of just *x*? How do we solve $3^{x-5} = 20$?

Well, we apply the same procedure, and then use the properties of equality for solving equations to isolate our variable *x*:

$$3^{x-5} = 20$$
$$x - 5 = \log_3 20$$
$$x = \log_3 20 + 5$$

and if we wished, we could use the base-change formula to get the decimal approximation

$$x = \frac{\log 20}{\log 3} + 5 \approx 5.72683$$

What about $3^{2x-1} = 20$?

$$3^{2x-1} = 20$$
$$2x - 1 = \log_3 20$$
$$2x = \log_3 20 + 1$$
$$x = \frac{\log_3 20 + 1}{2}$$

and once again we can use the base-change formula to get a decimal approximation if we wished.

Example

Solve $e^{7-x} = 3$.

Answer:

$$e^{7-x} = 3$$
$$7 - x = \ln 3$$
$$-x = \ln 3 - 7$$
$$x = -\ln 3 + 7$$
$$x = 7 - \ln 3$$
$$x \approx 5.90139$$

So 7 – ln 3 is the exact answer and 5.9 is a decimal approximation to one decimal place.

Example

Solve $5^{2x+6} = 152$.

Answer:

$$5^{2x+6} = 152$$
$$\log 5^{2x+6} = \log 152$$
$$(2x+6)\log 5 = \log 152$$
$$2x + 6 = \frac{\log 152}{\log 5}$$
$$2x = \frac{\log 152}{\log 5} - 6$$
$$x = \frac{1}{2}\left(\frac{\log 152}{\log 5} - 6\right)$$
$$x \approx -1.43924$$

Note also that you can check your answer by substituting *x* back into the original equation: $5^{2(-1.43924)+6}$, which your calculator should tell you is equal to 152.

Solving Exponential Equations Involving Coefficients

What if our equation has a coefficient in front of the log, as in our example below?

$$2e^{x+1} = 24$$

The easiest way to handle this is to divide both sides by 2 in order to reduce it to an equation that we know how to solve:

$$
\begin{aligned}
&2e^{x+1} = 24\\
&e^{x+1} = 12\\
&x+1 = \ln 12\\
&x = \ln 12 - 1\\
&x \approx 1.48491
\end{aligned}
$$

A common mistake you might be tempted to make is to try to change to the logarithmic form of the equation before dividing by two. This is tempting, but in order to do it properly, you need to use your properties of logarithms correctly:

$$
\begin{aligned}
&\ln\left(2e^{x+1}\right) = \ln 24\\
&\ln 2 + \ln e^{x+1} = \ln 24\\
&\ln e^{x+1} = \ln 24 - \ln 2\\
&x+1 = \ln 24 - \ln 2\\
&x = \ln 24 - \ln 2 - 1
\end{aligned}
$$

You might look at this and say that the solution is different from the previous method's answer, but remember the quotient rule: $\ln 24 - \ln 2 = \ln \frac{24}{2} = \ln 12$. So the second method gives exactly the same result as the first method. However, it's a bit trickier than the first because you need more properties of logs, so the first method would be the recommended one. In other words, if you can, then divide both sides by the coefficient on the exponential term to get the "bare exponential" on one side, and then change to the equivalent logarithmic equation.

Solving Logarithmic Equations

A logarithmic equation is (you've guessed it!) an equation with a logarithm in it. Our general method of solution will be to find out where the variable is. If the variable is buried within the logarithm, such as the example

$$\log_5(x+3) = 2$$

then our procedure will involve rewriting the equation into exponential form. For our example, this will be our next step:

$$
\begin{aligned}
&\log_5(x+3) = 2\\
&x+3 = 5^2
\end{aligned}
$$

and you can now see that this is a single equation in one variable and we will use our usual equation-solving steps to isolate *x*.

$$x+3=25$$
$$x=22$$

As our solution is a nice integer, we don't need to worry about decimal approximations for our answer.

What if there are logarithms on both sides of the equation? There is a property of logarithms similar to the one-to-one property of exponential functions. Namely, if $\log_a m = \log_a n$, then $m = n$ provided that both logarithms have the same base. Let's move to an example:

$$\log_5(2x-3)=\log_5(x+1)$$
$$2x-3=x+1$$
$$x=4$$

Example

Solve $\log_3 8 = \log_3(1-x)$.

Answer:

$$\log_3 8 = \log_3(1-x)$$
$$8=1-x$$
$$x=1-8$$
$$x=-7$$

Notice that the base is completely irrelevant to the solution, provided that it is the same on both sides.

What if the variable is buried within the base of the exponent, as in the following example?

$$\log_{2x} 36 = 1$$
$$36=(2x)^1$$
$$36=2x$$
$$x=18$$

Example:

Solve $\log_{x-1} 27 = 3$.

Answer:

$\log_{x-1} 27 = 3$

$27 = (x-1)^3$

$3 = x - 1$

$x = 4$

Note: if you're wondering how we got from the second step to the third step, we just took the cube root of both sides. The cube root of 27 is 3 and the cube root of anything cubed is just itself. It's a bit more tricky if we look at raising to an even power, but for the purposes of this class, we'll restrict ourselves to the odd powers for ease of use.

Section 5.4: Solving Equations with Exponents and Logs

Exercises

Solve. When appropriate, give your answer as a decimal approximation rounded to three places.

1. $5^x = 0.1$

2. $3^{2x} = 24$

3. $10^q = 15$

4. $2^{1-x} = 0.00004$

5. $6^{2x+1} = 36$

6. $7^{5-m} = 10$

7. $3^{x-6} = 1$

8. $25 = 0.5^y$

9. $1.02^x = 3$

10. $e^{-5t} = 0.25$

11. $e^{7t} = 3$

12. $1.01^{10x} = 5$

13. $5(1.015)^{8x} = 10$

14. $10e^{-5t} = 2$

15. $2(1.02)^{3t} = 8$

16. $1000e^{-10t} = 100$

Solve for *x*. Give exact answers.

17. $\log(1-x) = 1$

18. $\log(2x+1) = 0$

19. $\log_2(x-5)=3$

20. $\ln x=2$

21. $\log_x(3)=-1$

22. $\log_x(8)=3$

23. $\log_3(x-2)=-1$

24. $\log_5(x^3-2)=2$

25. $\log_x(8)=1/2$

26. $\log_{x+2}(64)=3$

27. $\log\left(x+5\right)=\log\left(7\right)$

28. $\ln\left(1-x\right)=\ln\left(9+x\right)$

29. $\log_2\left(x^3-1\right)=\log_2\left(26\right)$

30. $\log_{1.02}\left(x^2+5\right)=\log_{1.02}\left(x^2+x\right)$

31. $\ln\left(\frac{2}{3}x+1\right)=\ln\left(\frac{3}{2}x-4\right)$

32. $\log\left(0.01x\right)=\log\left(5\right)$

Section 5.4: Solving Equations with Exponents and Logs

Solutions

1. $$5^x = 0.1$$ $$x = \log_5 0.1 = \frac{\ln 0.1}{\ln 5} = -1.431$$

2. $$3^{2x} = 24$$ $$2x = \log_3 24$$ $$x = \frac{1}{2}\log_3 24 = \frac{\ln 24}{2\ln 3} = 1.446$$

3. $$10^q = 15$$ $$q = \log 15 = 1.176$$

4. $$2^{1-x} = 0.00004$$ $$1 - x = \log_2 0.00004$$ $$x = 1 - \log_2 0.00004 = 1 - \frac{\ln 0.00004}{\ln 2} = 15.610$$

5. $$6^{2x+1} = 36$$ $$6^{2x+1} = 6^2$$ $$2x + 1 = 2$$ $$2x = 1$$ $$x = 1/2 = 0.5$$

6. $$7^{5-m} = 10$$ $$5 - m = \log_7 10$$ $$m = 5 - \log_7 10 = 5 - \frac{\ln 10}{\ln 7} = 3.817$$

7. $$3^{x-6} = 1$$ $$3^{x-6} = 3^0$$ $$x - 6 = 0$$ $$x = 6$$

8.
$$25 = 0.5^y$$
$$\log_{0.5} 25 = y$$
$$y = \log_{0.5} 25 = \frac{\ln 25}{\ln 0.5} = -4.644$$

9.
$$1.02^x = 3$$
$$x = \log_{1.02} 3 = \frac{\ln 3}{\ln 1.02} = 55.478$$

10.
$$e^{-5t} = 0.25$$
$$-5t = \ln 0.25$$
$$t = -\frac{\ln 0.25}{5} = 0.277$$

11.
$$e^{7t} = 3$$
$$7t = \ln 3$$
$$t = \frac{\ln 3}{7} = 0.157$$

12.
$$1.01^{10x} = 5$$
$$10x = \log_{1.01} 5$$
$$x = \frac{\log_{1.01} 5}{10} = \frac{\ln 5}{10 \ln 1.01} = 16.175$$

13.
$$5(1.015)^{8x} = 10$$
$$(1.015)^{8x} = 2$$
$$8x = \log_{1.015} 2$$
$$x = \frac{\log_{1.015} 2}{8} = \frac{\ln 2}{8 \ln 1.015} = 5.819$$

14.
$$10e^{-5t} = 2$$
$$e^{-5t} = \frac{1}{5}$$
$$-5t = \ln(1/5)$$
$$t = -\frac{\ln(1/5)}{5} = 0.321$$

15.
$$
\begin{aligned}
&2(1.02)^{3t}=8\\
&(1.02)^{3t}=4\\
&3t=\log_{1.02}4\\
&t=\frac{\log_{1.02}4}{3}=\frac{\ln 4}{3\ln 1.02}=23.335
\end{aligned}
$$

16.
$$
\begin{aligned}
&1000e^{-10t}=100\\
&e^{-10t}=\frac{1}{10}\\
&-10t=\ln(1/10)\\
&t=-\frac{\ln(1/10)}{10}=0.230
\end{aligned}
$$

17.
$$
\begin{aligned}
&\log(1-x)=1\\
&1-x=10^1\\
&-x=9\\
&x=-9
\end{aligned}
$$

18.
$$
\begin{aligned}
&\log(2x+1)=0\\
&2x+1=10^0\\
&2x+1=1\\
&2x=0\\
&x=0
\end{aligned}
$$

19.
$$
\begin{aligned}
&\log_2(x-5)=3\\
&x-5=2^3\\
&x-5=8\\
&x=13
\end{aligned}
$$

20.
$$
\begin{aligned}
&\ln x=2\\
&x=e^2
\end{aligned}
$$

21.
$$\log_x(3) = -1$$
$$3 = x^{-1}$$
$$3 = \frac{1}{x}$$
$$x = \frac{1}{3}$$

22.
$$\log_x(8) = 3$$
$$8 = x^3$$
$$x = 2$$

23.
$$\log_3(x-2) = -1$$
$$x - 2 = 3^{-1}$$
$$x = 2 + \frac{1}{3}$$
$$x = \frac{7}{3}$$

24.
$$\log_5(x^3 - 2) = 2$$
$$x^3 - 2 = 5^2$$
$$x^3 = 27$$
$$x = 3$$

25.
$$\log_x(8) = 1/2$$
$$8 = x^{1/2}$$
$$x = 64$$

26.
$$\log_{x+2}(64) = 3$$
$$64 = (x+2)^3$$
$$4 = x + 2$$
$$x = 2$$

27.
$$\log\left(x+5\right) = \log\left(7\right)$$
$$x + 5 = 7$$
$$x = 2$$

28.

$$\begin{aligned}
&\ln(1-x)=\ln(9+x)\\
&1-x=9+x\\
&-2x=8\\
&x=-4
\end{aligned}$$

29.

$$\begin{aligned}
&\log_2\left(x^3-1\right)=\log_2(26)\\
&x^3-1=26\\
&x^3=27\\
&x=3
\end{aligned}$$

30.

$$\begin{aligned}
&\log_{1.02}\left(x^2+5\right)=\log_{1.02}\left(x^2+x\right)\\
&x^2+5=x^2+x\\
&5=x\\
&x=5
\end{aligned}$$

31.

$$\begin{aligned}
&\ln\left(\frac{2}{3}x+1\right)=\ln\left(\frac{3}{2}x-4\right)\\
&\frac{2}{3}x+1=\frac{3}{2}x-4 \quad \text{(multiply both sides by 6)}\\
&4x+6=9x-2 \qquad 4\\
&30=5x\\
&x=6
\end{aligned}$$

32.

$$\begin{aligned}
&\log(0.01x)=\log(5)\\
&0.01x=5\\
&x=500
\end{aligned}$$

Section 5.5: Applications

Now that we have the tools to solve equations in which the variable is in the exponent, we can look at a variety of applications that involve exponential equations. The three types that we will be examining are compound interest, exponential growth, and exponential decay.

Compound Interest Problems

We have seen previously that compound interest may be calculated by

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

where A is the amount of investment after t years, P is the principal (the original amount of the investment), r is the annual interest rate, n is the number of compounding periods per year, and t is the total time of investment.

If, instead, we look at interest compounded **continuously**, then the equation we are interested in is

$$A = Pe^{rt}$$

where A is the amount of the investment, P is the principal, r is the interest rate, and t is the time of the investment.

Example

Tony decides to put $2000 in a term deposit which is compounded monthly. If his term deposit has an annual interest rate of 2.4%, how long will it take him to double his money?

Answer:

Since Tony is doubling his money, $P = 2000$ and $A = 4000$. Expressed as a decimal, the interest rate r is 0.024. Putting this into the formula, we get

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$
$$4000 = 2000\left(1+\frac{0.024}{12}\right)^{12\times t}$$
$$2 = (1.002)^{12t}$$
$$\ln 2 = 12t\ln 1.002$$
$$t = \frac{\ln 2}{12\ln 1.002} = 28.91$$

Tony will double his money in just under 29 years.

Note that for this example, we didn't actually need to know the original amount. If instead all we knew is that Tony's money had doubled, we would still be able to solve the problem: the initial amount would be P, and the final amount A would just equal $2P$. Then when you divide both sides by P, the two Ps would cancel and you'd be left with the same calculation:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$
$$2P = P\left(1+\frac{0.024}{12}\right)^{12\times t}$$
$$2 = (1.002)^{12t}$$
$$\ln 2 = 12t\ln 1.002$$
$$t = \frac{\ln 2}{12\ln 1.002} = 28.91$$

which results in the same answer.

Exponential Growth Problems

Anything that undergoes exponential growth (computer viruses, mosquitoes, fluffy bunnies, the number of computers connected to the Internet) will obey the equation for exponential growth,

$$P = P_o e^{rt}$$

where P is the population at time t, P_o is the initial population (the population at time t=0), r is the exponential growth rate, and t is the elapsed time. You'll notice that this looks almost identical to the equation for continuous compounding, which is no accident since the two situations behave the same way. You may have seen this equation with k instead of r for the growth rate and with either A and A_o or N and N_o for the population variables. These are just conventions and it doesn't matter which version you use – the results should be the same.

For situations involving population growth, there are two ways of indicating how fast a particular population grows. You can either specify the growth rate r (expressed as a percentage per year – like an interest rate) or you can give another quantity called the **doubling time**. The doubling time is just the amount of time it takes for a population to grow to twice its original size (in fact, the previous example involving compound interest computed the doubling time for an investment). You can use this quantity to solve exponential growth problems, but it's usually a two-step process: you first have to find the growth rate r and then compute the quantity you're actually interested in.

Example

According to Wikipedia, 1.1 billion people currently enjoy regular access to the Web. If the annual growth rate of this population is 6.6%, how long will it take before twice as many people will have access to the Web?

Answer:

Instead of putting the billions of people into the exponential growth equation, I will simplify matters by just saying that $P = 2P_o$. Then

$$P = P_o e^{rt}$$
$$2P_o = Pe^{0.066t}$$
$$2 = e^{0.066t}$$
$$\ln 2 = 0.066t$$
$$t = \frac{\ln 2}{0.066} = 10.5$$

So the doubling time is 10.5 years.

Example

Brazil, on the other hand, has a doubling time of 7.7 years for people having regular access to the Web. If there are 5.6 million people currently connected, how many people will be connected in 10.5 years?

Answer:

Since the problem doesn't explicitly give us r, we'll have to calculate it first from the doubling time.

$$P = P_o e^{rt}$$
$$2P_o = P_o e^{r\,7.7}$$
$$2 = e^{7.7r}$$
$$\ln 2 = 7.7r$$
$$r = \frac{\ln 2}{7.7} = 0.090019 = 0.09$$

Brazil's growth rate is 9%. Now we'll put this back into our equation, using a time of 10.5 years. We could either put in 5,600,000 as our original population, or as I've done below, we could just leave our answer in millions of people.

$$P = P_o e^{rt}$$
$$P = 5.6e^{0.09 \times 10.5}$$
$$P = 14.4$$

Brazil will have 14.4 million people with regular access to the Web in 10.5 years.

Exponential Decay Problems

There are certain physical phenomena that undergo exponential decay: the rate of cooling or warming of objects at one temperature when moved to surroundings of different temperature, rates of chemical reactions, atmospheric pressure, and the amount of charge on a capacitor, to name a few. For the purposes of this class, we'll concentrate on examples involving radioactive decay. The equation for exponential decay looks very similar to the corresponding equation for exponential growth, with the exception that the exponent is negative:

$$P = P_o e^{-rt}$$

where P is the amount of the radioactive substance at time t, P_o is the initial amount of the substance (the amount at time t=0), r is the exponential decay rate, and t is the elapsed time. You will often see this equation using A and A_o or N and N_o for the amount, but for simplicity I will stick with the P notation.

It's also acceptable to use the original equation, $P = P_o e^{rt}$, for exponential decay. The only difference is that when you solve for r, you'll find that it's a **negative** quantity, so the end result looks just like the equation above containing the negative sign. It does not matter which convention you use, but you have to stick with the same convention throughout any one problem.

When a radioactive substance decays, how fast it is decaying is usually described by the **half-life**, the time it takes for only half of the original substance to be left. Once this is given, then the decay rate r may be calculated as before as an intermediate step in solving the problem.

Example

Carbon-14 has a half-life of 5730 years. If there was originally 15.0 grams of carbon-14, how much will be left after 1000 years? How much will have decayed away?

Answer:

First, we'll need to calculate r for $P = \frac{1}{2} P_o$:

$$P = P_o e^{-rt}$$
$$\frac{1}{2}P_o = P_o e^{-r5730}$$
$$\frac{1}{2} = e^{-5730r}$$
$$\ln\frac{1}{2} = -5730r$$
$$r = -\frac{\ln(1/2)}{5730} = 0.000121$$

Now that we've found r, we'll find P after 1000 years:

$$P = P_o e^{-rt}$$
$$P = 15.0e^{-0.000121\times 1000}$$
$$P = 13.3$$

There will be 13.3 grams of carbon-14 left.

To find the amount decayed away, subtract the amount left from the initial amount, so (15.0 – 13.3) grams, or 1.7 grams will have decayed away.

Section 5.5: Applications

Exercises

Solve the following compound interest problems.

1. If \$500 is deposited into an account paying 8% compounded quarterly, then how long will it take before the account contains \$700?

2. How long will it take to double an investment which is compounded continuously at an interest rate of 5% per year?

3. Charlotte invests a certain amount of money at 12% per year, compounded semi-annually. How long will it take to double her initial investment?

4. Doug invests a certain amount of money at 12% per year, compounded monthly. After five years, his investment is worth \$2500. How much did he invest initially?

5. Steve invests \$500 in an account that compounds continuously. If the account is worth \$727.50 after five years, what was the interest rate on Steve's account?

6. As part of a science experiment, Pat is cryogenically frozen for many years. Once she wakes up, she finds that her bank account, which had a balance of \$1600 when she was frozen, has grown to 5.8 million dollars. The interest rate on her account has remained at 6% the entire time. How long has she been asleep if her account was compounded

   a) monthly?
   b) continuously?

Solve the following exponential growth problems.

7. The Saanich News reported on page A3 of the edition printed Wednesday, November 14, 2007 that UVic has a rabbit problem. The article states that "If 95 percent of the University of Victoria's bunnies were blown off the face of the planet, they would repopulate to their current population within two years". What is the doubling time for fluffy bunnies at UVic?

8. Could there have been vampires in Transylvania? Assume that each vampire creates another vampire by biting a human during the full moon. Starting then with one vampire, how long would it take before every person in Transylvania (estimated population of 2.4 million people) was turned into a vampire? (Assume no population grown in Transylvania during this time.)

9. Barry Hendy of Kodak Australia has found that the number of pixels you can get per dollar for a digital camera grows exponentially. His graph on the Wikipedia commons (http://en.wikipedia.org/wiki/Image:Hendys_Law.jpg) shows that in 1999

you could get about 1000 pixels per dollar, and in 2004 you could get 10,000 pixels per dollar. If this relationship is true, when could you get roughly 5000 pixels per dollar?

10. According to Wikipedia, the United Nations estimated the world's population to be about 6.070 billion people in the year 2000, growing at a rate of 1.14% per year.

    a) If that estimate is accurate, how many people are added to the Earth's population in a single year? In a single day?
    b) What's the doubling time for the world's population if the rate of growth remains constant?
    c) How many people will there be in 2050?

Solve the following exponential decay problems.

11. The fictional radioactive isotope Adamantium-354 has a half-life of 3.5 seconds. If you have exactly 1 gram of the substance to begin with, how much will be left after exactly one second?

12. After five days, the amount of Unobtainium-157 has decayed to 35% of the original amount. What is the half-life of this radioactive isotope?

13. If 80% of a radioactive element remains radioactive after 250 million years, then what percent remains radioactive after 600 million years? What is the half-life of this element?

14. The half-life of U-235 (uranium-235) is 7.0×10^8 years. After one billion years, how much of the original uranium-235 will remain? How much will have decayed away?

15. Dr. Evil, in his underground volcano lair, has manufactured a certain amount of the radioactive isotope Explodium-337, which has a half-life of 36 minutes. To save the day, after the isotope was made Austin Powers managed to create chaos and distract everyone for exactly two and a half hours. After that time, what fraction of the original Explodium is left?

16. In Smallville, Lex Luthor is threatening to destroy Superman's superpowers using Red Kryptonite. However, Lex's plans have misfired – Red Kryponite is radioactive and after 11.5 days, 88% has decayed away, leaving only 12% of the original material. What is the half-life of Red Kryptonite?

Section 5.5: Applications

Solutions

1.

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$
$$700 = 500\left(1+\frac{0.08}{4}\right)^{4t}$$
$$\frac{7}{5} = (1.02)^{4t}$$
$$\ln\left(\frac{7}{5}\right) = \ln(1.02)^{4t}$$
$$\ln\left(\frac{7}{5}\right) = 4t\ln 1.02$$
$$t = \frac{\ln(7/5)}{4\ln 1.02} = 4.2478$$

The account will contain $700 in $4\frac{1}{4}$ years.

2.

$$A = Pe^{rt}$$
$$2A_0 = Pe^{0.05t}$$
$$2 = e^{0.05t}$$
$$\ln 2 = 0.05t$$
$$t = \frac{\ln 2}{0.05} = 13.8629$$

The investment will double in 13.9 years. (Okay to round to 14.)

3.

solution #1:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$
$$2P = P\left(1+\frac{0.12}{2}\right)^{2\times t}$$
$$2 = (1.06)^{2t}$$
$$\log_{1.06} 2 = 2t$$
$$t = \frac{\log_{1.06} 2}{2} = \frac{\ln 2}{2\ln 1.06} = 5.94783$$

solution #2:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$
$$2P = P\left(1+\frac{0.12}{2}\right)^{2\times t}$$
$$2 = (1.06)^{2t}$$
$$\ln 2 = \ln(1.06)^{2t}$$
$$\ln 2 = 2t\ln 1.06$$
$$t = \frac{\ln 2}{2\ln 1.06} = 5.94783$$

Either solution gives the same answer: Charlotte's investment will double in just under 6 years.

4.
$$A = P\left(1+\frac{r}{n}\right)^{nt}$$
$$2500 = P\left(1+\frac{0.12}{12}\right)^{12\times 5}$$
$$2500 = P(1.01)^{60}$$
$$P = \frac{2500}{1.01^{60}} = 1376.12$$

Doug invested $1376.12 initially.

5.
$$A = Pe^{rt}$$
$$727.50 = 500e^{r5}$$
$$1.455 = e^{5r}$$
$$\ln 1.455 = 5r$$
$$r = \frac{\ln 1.455}{5} = 0.075001$$

The interest rate on Steve's account is 7.5%.

6. a) monthly:
$$A = P\left(1+\frac{r}{n}\right)^{nt}$$
$$5\ 800\ 000 = 1600\left(1+\frac{0.06}{12}\right)^{12t}$$
$$3625 = (1.005)^{12t}$$
$$\ln 3625 = \ln(1.005)^{12t}$$
$$\ln 3625 = 12t\ln 1.005$$
$$t = \frac{\ln 3625}{12\ln 1.005} = 136.935$$

b) continuously:
$$A = Pe^{rt}$$
$$5{,}800{,}000 = 1600e^{0.06t}$$
$$3625 = e^{0.06t}$$
$$\ln 3625 = 0.06t$$
$$t = \frac{\ln 3625}{0.06} = 136.593$$

So, in either case, Pat was cryogenically frozen for 137 years.

7. For this problem, you are starting with 0.05 of the population and after two years, getting the full 100% or just 1 times the population back.

find r:

$$\begin{aligned} P &= P_o e^{rt} \\ P &= 0.05Pe^{r(2)} \\ 20 &= e^{2r} \\ \ln(20) &= 2r \\ r &= \frac{\ln(20)}{2} \\ r &= 1.49787 \end{aligned}$$

find t:

$$\begin{aligned} P &= P_o e^{rt} \\ 2P_o &= P_o e^{1.49787t} \\ 2 &= e^{1.49787t} \\ \ln(2) &= 1.49787t \\ t &= \frac{\ln(2)}{1.49787} \\ t &= 0.462756 \end{aligned}$$

The doubling time for fluffy bunnies at UVic is 0.46 years (just over 5½ months).

8. If each vampire creates another vampire during the full moon, then the doubling time for the vampire population is one month. You can either do this problem in months, if you wish, remembering that your final answer will also be in months, or you can choose another unit providing that you are consistent. I will choose years for my answer, assuming 12 full moons per year (not totally accurate, but will get approximately the same answer):

find r:

$$\begin{aligned} P &= P_o e^{rt} \\ 2P_o &= P_o e^{r\left(\frac{1}{12}\right)} \\ 2 &= e^{r/12} \\ \ln(2) &= \frac{r}{12} \\ r &= 12\ln(2) \\ r &= 8.31777 \end{aligned}$$

find t:

$$\begin{aligned} P &= P_o e^{8.31777t} \\ 2,400,000 &= 1e^{8.31777t} \\ 2,400,000 &= e^{8.31777t} \\ \ln(2,400,000) &= 8.31777t \\ t &= \frac{\ln(2,400,000)}{8.31777} \\ t &= 1.76622 \end{aligned}$$

It would take 1.77 years (just over 1 ¾ years) to turn everyone into a vampire in Transylvania. I would have to conclude that either there were no vampires in Transylvania or their vampire hunters were very good indeed!

9. For this problem, I will take my initial P_o as 1000. Then my time will be measured from 1999, so for P = 10,000 in 2004 my t will be 5 years.

find r:

$$\begin{aligned} P &= P_o e^{rt} \\ 10,000 &= 1000e^{r5} \\ 10 &= e^{5r} \\ \ln(10) &= 5r \\ r &= \frac{\ln(10)}{5} \\ r &= 0.460517 \end{aligned}$$

find t:

$$\begin{aligned} P &= P_o e^{rt} \\ 5000 &= 1000e^{0.460517t} \\ 5 &= e^{0.460517t} \\ \ln(5) &= 0.460517t \\ t &= \frac{\ln(5)}{0.460517} \\ t &= 3.49485 \end{aligned}$$

Then you could get 5000 pixels 3.5 years after the initial time, so somewhere in 2002 to 2003. (The point is that it's not **halfway** in time between 1999 and 2004.)

10\. a) In one year, $\boxed{\begin{aligned}&P = P_o e^{rt}\\&P = 6.070\times10^9 e^{0.0114(1)}\\&P = 6.13959\times10^9\end{aligned}}$ so the change in population is (6.140 – 6.070) billion people = 0.07 billion people = 70 million people. In one day, $\boxed{\begin{aligned}&P = P_o e^{rt}\\&P = 6.070\times10^9 e^{0.0114(1/365)}\\&P = 6.07019\times10^9\end{aligned}}$ so the change in population is 0.00019 billion people, or 0.19 million people, or 190,000 people in one day. That's a lot of people!

b) $\boxed{\begin{aligned}&P = P_o e^{rt}\\&2P_o = P_o e^{0.0114t}\\&2 = e^{0.0114t}\\&\ln(2) = 0.0114t\\&t = \frac{\ln(2)}{0.0114}\\&r = 60.8\end{aligned}}$ The doubling time is 60.8 years.

c) $\boxed{\begin{aligned}&P = P_o e^{rt}\\&P = 6.070\times10^9 e^{0.0114(50)}\\&P = 1.0733\times10^{10}\end{aligned}}$ There will be 10.73 billion people in 2050, assuming that the rate of growth remains constant.

11\. find r: $\boxed{\begin{aligned}&P = P_o e^{-rt}\\&\frac{1}{2}P_o = P_o e^{-r(3.5)}\\&\frac{1}{2} = e^{-3.5r}\\&\ln\left(\frac{1}{2}\right) = -3.5r\\&r = \frac{\ln(1/2)}{-3.5} = 0.198042\end{aligned}}$ find A: $\boxed{\begin{aligned}&P = P_o e^{-rt}\\&P = 1e^{-0.198042(1)}\\&P = 0.820335\end{aligned}}$ There will be 0.82 g left.

12. find r:

$$\begin{aligned}
P &= P_o e^{-rt} \\
0.35P_o &= P_o e^{-r(5)} \\
0.35 &= e^{-5r} \\
\ln(0.35) &= -5r \\
r &= \frac{\ln(0.35)}{-5} \\
r &= 0.209964
\end{aligned}$$

find t:

$$\begin{aligned}
P &= P_o e^{-rt} \\
\frac{1}{2}P_o &= P_o e^{-0.209964t} \\
\frac{1}{2} &= e^{-0.209964t} \\
\ln\left(\frac{1}{2}\right) &= -0.209964t \\
t &= \frac{\ln(1/2)}{-0.209964} \\
t &= 3.30126
\end{aligned}$$

The halflife is 3.3 days.

13. I'm going to take my unit as millions of years.

find r:

$$\begin{aligned}
P &= P_o e^{-rt} \\
0.8A_o &= P_o e^{-r(250)} \\
0.8 &= e^{-250r} \\
\ln(0.8) &= -250r \\
r &= \frac{\ln(0.8)}{-250} \\
r &= 0.000892574
\end{aligned}$$

find A:

$$\begin{aligned}
P &= P_o e^{-rt} \\
P &= P_o e^{-0.000892574(600)} \\
P &= P_o(0.585) \\
P &= 0.585P_o
\end{aligned}$$

find t:

$$\begin{aligned}
P &= P_o e^{-rt} \\
\frac{1}{2}P_o &= P_o e^{-0.000892574t} \\
\frac{1}{2} &= e^{-0.000892574t} \\
\ln\left(\frac{1}{2}\right) &= -0.000892574t \\
t &= \frac{\ln(1/2)}{-0.000892574} \\
t &= 776.57
\end{aligned}$$

There will be 58.5% left after 600 million years, and the halflife is 780 million years.

14. If you wish, you may use billions of years as your unit. I've chosen to use years, myself.

find r:

$$P = P_o e^{-rt}$$
$$\frac{1}{2} P_o = P_o e^{-r\left(7.0\times10^8\right)}$$
$$\frac{1}{2} = e^{-7.0\times10^8 r}$$
$$\ln\left(\frac{1}{2}\right) = -7.0\times10^8 r$$
$$r = \frac{\ln(1/2)}{-7.0\times10^8}$$
$$r = 9.9021\times10^{-10}$$

find A:

$$P = P_o e^{-rt}$$
$$P = P_o e^{-9.9021\times10^{-10}\left(1\times10^9\right)}$$
$$P = 0.371499 P_o$$

37% of the original uranium will remain, so 63% must have decayed away.

15. find r:

$$P = P_o e^{-rt}$$
$$\frac{1}{2} P_o = P_o e^{-r(36)}$$
$$\frac{1}{2} = e^{-36r}$$
$$\ln\left(\frac{1}{2}\right) = -36r$$
$$r = \frac{\ln(1/2)}{-36} = 0.019254$$

find P:

$$P = P_o e^{-rt}$$
$$P = P_o e^{-0.019265(2.5\times 60)}$$
$$P = 0.055681 P_o$$

Only 5.6% of the original amount of Explodium will be left.

16. find r:

$$P = P_o e^{-rt}$$
$$0.12 P_o = P_o e^{-r(11.5)}$$
$$0.12 = e^{-11.5r}$$
$$\ln(0.12) = -11.5r$$
$$r = \frac{\ln(0.12)}{-11.5}$$
$$r = 0.184371$$

find t:

$$P = P_o e^{-rt}$$
$$\frac{1}{2} P_o = P_o e^{-0.184371t}$$
$$\frac{1}{2} = e^{-0.184371t}$$
$$\ln\left(\frac{1}{2}\right) = -0.184371t$$
$$t = \frac{\ln(1/2)}{-0.184371}$$
$$t = 3.75953$$

The halflife is 3.8 days.