Section 5.1: Exponential Functions

We should begin by defining exponential functions. Recall that a function is a relation in which for every *x*-value, there is only one *y*-value. An exponential function, then, is a function of the form

$$y = a^x$$

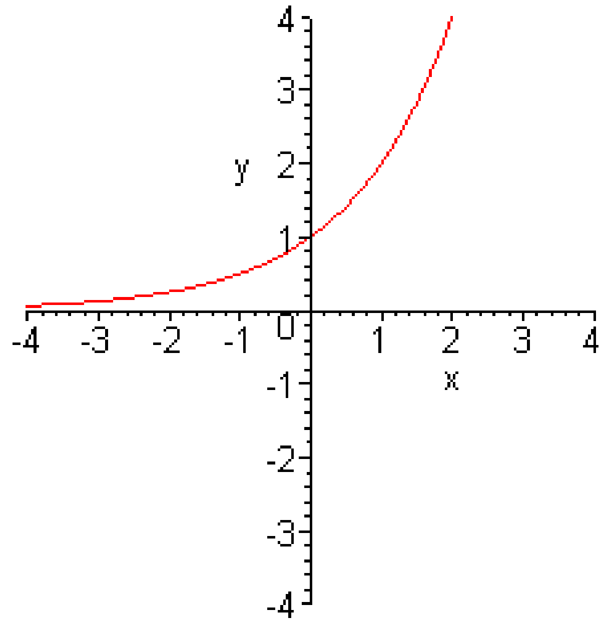
where the variable *x* is in the exponent. The constant *a* is then a real number with the restrictions that $a > 0$ and $a \neq 1$.

(We want $a > 0$ so that the graph of $y = a^x$ is a nice smooth curve – if we tried to graph $y = (-2)^x$, the graph would oscillate between positive and negative values since $(-2)^2$ equals +4 and $(-2)^3$ equals –8. The $a \neq 1$ restriction will be explained when we look at solving equations.)

Let's start by graphing $y = 2^x$. The brute force method is to draw up a table of values.

x	$y = 2^x$
-3	1/8
-2	¼
-1	½
0	1
1	2
2	4
3	8

By plotting these points and drawing a nice, smooth curve, we'll get a graph that looks like the following.



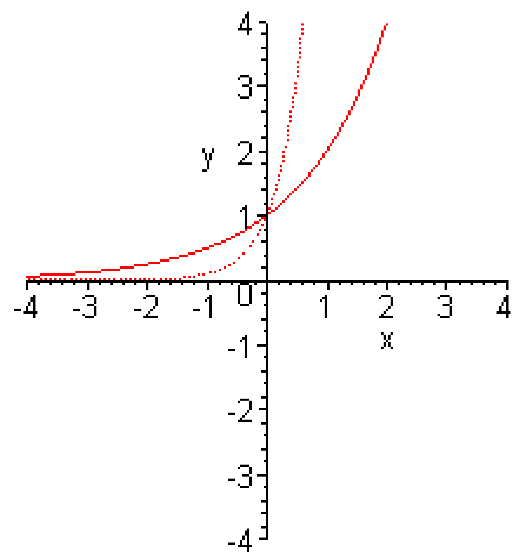
There are a few things to notice from this graph.

1. Note that the graph climbs <u>very</u> rapidly as *x* gets larger (this is what we really mean when we speak of "growing exponentially").

2. As *x* gets smaller (more negative), observe that *y* gets very small indeed but never touches the *x*-axis (we say that it approaches the axis asymptotically).

3. Notice that all values of *x* are allowed in the exponent, but the resulting *y* just be positive – never negative and never zero.

What if we were to graph $y=10^x$ on the same graph? We'd get



where the solid line is $y = 2^x$ and the dotted line is $y = 10^x$.

The point (0,1) is common to both graphs: $y = a^x$ will contain the point (0,1) since $a^0 = 1$ for all a. The $y = 10^x$ line will simply rise more steeply as x gets larger, and will approach the x-axis more quickly as x gets more negative.

Solving Exponential Equations

We will start by solving a particular type of exponential equation in which you can match the bases. There is a nice property of exponential functions that will come in handy.

One-to-One Property of Exponential Functions

For $a > 0$ and $a \neq 1$,

if $a^m = a^n$, then $m = n$.

(Here's why we want $a \neq 1$: if $a = 1$, then $a^m = 1$ for all m and we can't find a unique answer for m.)

What this means is that if we can match the bases on each side of an equation, then we can match the exponents as well.

Example

Solve $3^{x+1} = 81$.

Answer: We start by trying to write each side using the same base. In this case, we can do this by substituting 3^4 for 81 on the right-hand side:

$3^{x+1} = 3^4$

If the bases are the same, then the exponents must also be the same. So,

$x + 1 = 4$

$x = 3$

So our solution set is {3}, and if we substitute it back into the original equation, we see that the left-hand side is $3^{3+1} = 3^4 = 81$. Check!

Example

Solve $2^x = \frac{1}{4}$.

Answer:

$$2^x = \frac{1}{4}$$
$$2^x = 2^{-2}$$
$$x = -2$$

And the solution set is {-2}.

Example

Solve $\left(\frac{1}{5}\right)^x = 125$.

Answer:

$$\left(\frac{1}{5}\right)^x = 125$$
$$\left(5^{-1}\right)^x = 5^3$$
$$5^{-x} = 5^3$$
$$-x = 3$$
$$x = -3$$

And the solution set is {-3}.

Example

Solve $16^x = 4$.

Answer:

$$16^x = 4$$
$$\left(4^2\right)^x = 4^1$$
$$4^{2x} = 4^1$$
$$2x = 1$$
$$x = 1/2$$

And the solution set is { ½ }.

Compound Interest

If a loan, an investment, or a mortgage is calculated with **compound interest**, that means that after a certain period (called the compounding period), interest is deposited into the account, and then interest is paid on that interest. Let's start out with principal P compounded annually with interest rate r. After one year, the amount A in the account will be the principal P plus interest $P \cdot r$, or $A = P(1+r)$. After the second year, then the new amount will be $P(1+r)^2$, and after t years, the total amount $A = P(1+r)^t$.

To generalize this to **any** compounding period, the interest rate for that compounding period will be r/n, where n is now the number of compounding periods per year. So our formula becomes

$$A = P\left(1+\frac{r}{n}\right)^{nt},$$

where A is the amount of investment after t years, P is the principal (the original amount of the investment), r is the annual interest rate, n is the number of compounding periods per year, and t is the total time of investment.

Example

If $1000 is deposited into an account paying 10% per year compounded monthly, how much will be in the account after 10 years?

Answer:

$P = 1000$, $r = 0.1$, $n = 12$, and $t = 10$

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 1000\left(1+\frac{0.1}{12}\right)^{12\times 10} \\ &= 2707.04 \end{aligned}$$

So the account will contain the amount $2707.04 after ten years.

Let's see how the amount after investment varies as a function of the compounding period. Consider the following table.

	A	B	C	D
1	*compounding period*		$1000\left(1+\frac{0.1}{x}\right)^{(10\,x)}$	
2	*yearly*	1	2593.7424601000	
3	*monthly*	12	2707.0414908622531148	
4	*weekly*	52	2715.6726950308601142	
5	*daily*	365	2717.9095545777538264	
6	*hourly*	8760	2718.2663133141708018	
7	*every minute*	525600	2718.2815698708146523	
8	*every second*	31536000	2718.2818241694182054	
9				
10	*continuously*		2718.2818284590452354	

You can see that when the compounding period gets sufficiently small, the amount gets closer and closer to a particular value. This value is the continuous limit, and can be given by the equation

$$A = Pe^{rt}$$

where A is the amount of the investment, P is the principal, r is the interest rate, and t is the time of the investment. The number e is a constant, equal to 2.71828182845904… (which is the continuous limit above divided by our initial investment of $1000). This number e is irrational, so when you represent it by a decimal, the decimal equivalent does not repeat and does not terminate (exactly like the decimal equivalent of π).

The equation for the continuous limit does not just apply to investments – it is the equation for all types of exponential growth. If you have a population of mosquitoes or fluffy bunnies or bacteria and that population has enough resources to grow unchecked, then it will grow according to that equation, in other words **exponentially**.