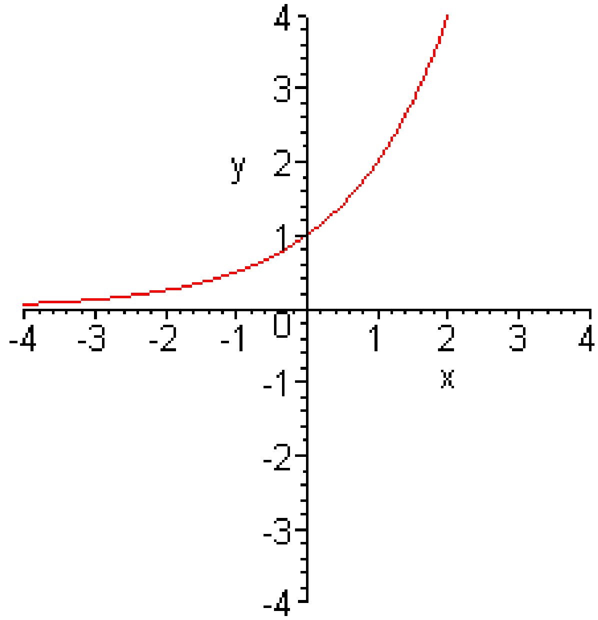
Section 5.2: Logarithmic Functions

Let's look at the $y = 2^x$ graph from the last section:



You'll notice that we plotted points for **integer** values of *x*, which led to either integer values for *y* or some familiar fractions like ½, ¼ and so on. However, when we drew the curved line through the points, we **implied** that *x* could take on any real values, including fractions, decimals or irrational numbers.

For example, in the graph above when *x* = 0.8, what's the corresponding value of *y*? We can draw a line upwards from the *x*-axis at *x* = 0.8 to find that it intersects the curve when *y* is approximately equal to 1.7. This means that $2^{0.8} \approx 1.7$ (the squiggly equals-sign means "approximately equal to"). We'll find out later how to calculate this exactly.

Also, we can ask, "For what value of *x* does $2^x = 3$?" From the graph, we see that when *y* = 3, the *x*-value is about 1.6 or so. The question, then, is how can we perform these calculations exactly instead of estimating from a graph? In the last section, you'll recall that we did many examples of solving similar equations like $2^x = 4$ exactly. The technique we used was to rewrite the right-hand side so that the bases matched. However, we can't use that technique here, because we do not know how to write the number 3 as 2 raised to some integer or fraction.

Instead, we have to use a new function: the **logarithm**.

Inverses

Before we examine the logarithm, we should talk first about functions and their inverses. For example, let's take the function $y = 2x + 4$. If for some strange reason, we wanted to solve for *x* instead, we'd perform the following procedure:

$$\begin{aligned} y &= 2x + 4 \\ y - 4 &= 2x \\ \frac{y-4}{2} &= \frac{2x}{2} \\ \frac{1}{2}y - 2 &= x \\ x &= \frac{1}{2}y - 2 \end{aligned}$$

and then $x = \frac{1}{2}y - 2$ is the inverse to our original function. Let's find some others: if $y = x^3$, then $x = \sqrt[3]{y}$ is the inverse. If $y = \sin x$, then $x = \sin^{-1} y$. The logarithm function is then the inverse to the exponential function: it's what we get if we solve the equation $y = a^x$ for *x*.

Logarithms

Let's present the exponential equation along with its inverse:

$$\begin{aligned} y &= a^x \\ \log_a y &= x \end{aligned}$$

These two equations are inverses of each other. To get to the second (scary-looking) one from the first, move the *x* in the exponent down so it's by itself on the right-hand side. The base, *a*, then moves to the other side and becomes the base of the logarithm, and the *y* is then the argument of the logarithm (what the logarithm is operating on). This probably doesn't seem particularly enlightening, so let's move to a numerical example.

Example

Solve the following equation for *x*: $10^x = 3$. Give both an exact answer and a decimal approximation to two decimal places.

Answer:

We want to move the base, 10, to the other side as the base of the logarithm. We then get:

$$10^x = 3$$
$$x = \log_{10} 3$$

This is the exact answer, $x = \log_{10} 3$. To calculate it exactly, you'll want a scientific calculator with a log button. If you enter "log 3" (on some calculators, you'll enter 3 and then hit log), then you'll get 0.477121 with a few more decimals. Rounding to two decimal places gives x = 0.48. The log button on scientific calculators means "$\log_{10}$". We'll find out how to calculate logs with different bases later.

How can we check? Our original equation was $10^x = 3$. By plugging $10^{0.477121}$ into the calculator, we should get 3. (If you use the rounded-off answer of 0.48, you'll get a number that's a little more than 3, 3.01995.) So we can see that we have arrived at the decimal approximation to our original equation.

Note, also, that we should expect an answer somewhere between zero and one. Why? Well, 10^0 is equal to 1 and 10^1 = 10, so finding the exponent that will give an answer of 3 must lie between 0 and 1.

Example

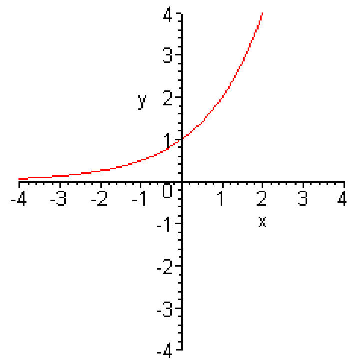
Solve the following equation for x: $10^x = 4200$. Give both an exact answer and a decimal approximation to two decimal places.

Answer:

$$10^x = 4200$$
$$x = \log_{10} 4200$$
$$x \approx 3.62$$

Again, we should expect an answer between 3 and 4, because 10^3 = 1000 and 10^4 = 10,000.

Now, logs can have any base that obeys the rules we studied for bases of exponents in the last section, namely that our base a must be greater than zero and not equal to one. But there are some properties that we should be aware of. Let's look once more at the $y = 2^x$ graph.



Notice that we can plug any real number into *x*, the exponent of $y = 2^x$, but the resulting *y* is always positive. For the inverse function, we are putting in the *y*-value and finding the resulting exponent. So the logarithm function $\log_a y = x$ can only having positive values for *y* applied to it, but the resulting *x* can be any real number.

Example

Calculate $\log_{10} 100$, $\log_{10} 0.1$, $\log_{10} 0$, and $\log_{10}(-10)$ using your calculator.

Answer:

Remembering that the "log" button on your calculator finds logs of base 10, you can just type in the above numbers. Your calculator should find that $\log_{10} 100 = 2$. This should make sense, because if you set $\log_{10} 100$ to a variable and then change the equation into exponential form, you get

$$x = \log_{10} 100$$
$$10^x = 100$$

But 100 is just 10 squared, so *x* must equal 2, using our matching-bases technique from the previous section.

Using the calculator for $\log_{10} 0.1$, you should get -1 (which should make sense because 10^{-1} = 0.1. However, trying to calculate $\log_{10} 0$ and $\log_{10}(-10)$ should give you errors on your calculator. That is because the *y*-value of the logarithm function $\log_a y$ must always be positive.

Natural Logs and Common Logs

Although logarithms can have any base provided that it's positive and not equal to one, the two most common bases are 10 and *e*. Because they are used so frequently, there are some conventions we should examine. You've already notice one: even though the button on your calculator calculates logs with base of 10, it doesn't have the number 10 explicitly written. That's because the convention is that when the base is missing, the base meant is 10, and the logarithm with base 10 is called the **common log**.

(It's like the convention that the variable x is actually x^1, but we commonly omit the exponent if it's equal to one. Also, for radicals, if the index n is missing, then it's assumed to be equal to 2: $\sqrt{x} = \sqrt[2]{x}$. If we are taking some other root, like the cube root, we have to remember to write the index for the radical sign.)

To write logarithms with base of e, we replace the "$\log_e$" by just "ln", since the log with base e is called the **natural log**. So $\log_e x = \ln x$, and your calculator should also have a button with "ln" on it.

Example

Solve the following equation for x: $e^x = 24$. Give both an exact answer and a decimal approximation to two decimal places.

Answer:

$$
\begin{aligned}
e^x &= 241 \\
x &= \ln 241 \\
x &\approx 5.48
\end{aligned}
$$

Example

Solve the following equation for x: $e^{2x+7} = 24$. Give both an exact answer and a decimal approximation to two decimal places.

Answer:

$$
\begin{aligned}
e^{2x+7} &= 24 \\
2x+7 &= \ln 24 \\
2x &= \ln 24 - 7 \\
x &= \frac{\ln 24 - 7}{2} \\
x &\approx -1.91
\end{aligned}
$$

So $x = \frac{\ln 24 - 7}{2}$ is the exact answer and –1.91 is the approximate answer.

Exact Calculations

Let's do a little more work on understanding what the various parts of the logarithm mean. For example, what does $\log_2 8$ mean?

$$x = \log_2 8$$
$$2^x = 8$$
$$2^x = 2^3$$
$$x = 3$$

So $\log_2 8$ means "what do you have to raise 2 to the power of to get 8?"

Example

Simplify the following:

a) $\log_3 27$

b) $\log_{10} \frac{1}{100}$

c) $\log_6 1$

d) $\log_5 0$

Answer:

a) $3^3 = 27$, so we must raise 3 to the power of **3** to get 27, so the logarithm equals 3.

b) $10^{-2} = \frac{1}{100}$, so $\log_{10} \frac{1}{100} = -2$.

c) $6^0 = 1$, so $\log_6 1 = 0$.

d) You can't put non-positive numbers into logarithms, so $\log_5 0$ is undefined.

Example

Simplify the following:

a) $\log_a a^4$

b) $\log_a a$

c) $\log_a 1$

Answer:

a) To get a^4, we must raise *a* to the fourth power, so $\log_a a^4$ must be 4.

b) $a^1 = a$, so $\log_a a = 1$.

c) $a^0 = 1$, so $\log_a 1 = 0$.

Logarithmic vs. Exponential Forms

As we have seen, every exponential equation has a corresponding logarithmic equation. We should practice transforming between the two forms.

Example

Write the corresponding exponential/logarithmic forms for the following equations. Don't simplify or solve!

a) $a = \log_b c$

b) $q = m^n$

Answer

a) This equation is in logarithmic form. To get it to exponential form, move the base to the other side of the equation, and the variable that's there already gets bumped up into the exponent: $b^a = c$.

b) This equation is in exponential form. To convert it to logs, the base *m* moves to the other side and becomes the base of the logarithm, $\log_m q = n$.