## Section 5.3: Properties of Logarithms

Let's examine some properties of logarithms that will allow us to solve equations containing logs more easily.

**The Product Rule**

Let's start with a numerical example to develop the ideas behind the product rule of logarithms. First, let's calculate the following logarithms (using a calculator!), remembering that "log" means "$\log_{10}$".

$$\begin{aligned}&\log 5.7\\&\log 57\\&\log 570\\&\log 5700\end{aligned}$$

We find that

$$\begin{aligned}\log 5.7 &= 0.755875\\ \log 57 &= 1.755875\\ \log 570 &= 2.755875\\ \log 5700 &= 3.755875\end{aligned}$$

and when we look at the numbers, we can see a pattern developing. We can rewrite the numbers on the right-hand side to get

$$\begin{aligned}\log 57 &= 1+0.755875\\ \log 570 &= 2+0.755875\\ \log 5700 &= 3+0.755875\end{aligned}$$

which may not seem particularly enlightening until we remember that $1=\log 10$ and notice also that 0.755875 is just log 57. We then get

$$\begin{aligned}\log 57 &= \log 10+\log 5.7\\ \log 570 &= \log 100+\log 5.7\\ \log 5700 &= \log 1000+\log 5.7\end{aligned}$$

and finally that

$$\begin{aligned}\log(10\times 5.7) &= \log 10+\log 5.7\\ \log(100\times 5.7) &= \log 100+\log 5.7\\ \log(1000\times 5.7) &= \log 1000+\log 5.7\end{aligned}$$

In other words, if you are taking the logarithm of a product, it's equal to the sum of the logs of the individual terms of the product. Writing that in symbols, we get

$$\log_a(MN) = \log_a M + \log_a N .$$

Note, however, that if you wish to combine two logarithms into a single one using this property, the **bases must be the same**!

***Example***

Express as a single logarithm:

a) $\log x + \log y$

b) $\ln 2 + \ln 3$

c) $\log_b x^2 + \log_b x^3$

d) $\log 24x^2 + \log \frac{x}{8}$

Answers:

a) $\log xy$

b) $\ln 6$

c) $\log_b x^5$

d) $\log 3x^3$

***Example***

Use the product rule to write an equivalent expression for the following:

a) $\log_b 2x$

b) $\ln mn$

c) $\log 6p$

d) $\log_b 3pq$

Answers:

a) $\log_b 2 + \log_b x$

b) $\ln m + \ln n$

c) $\log 6 + \log p$

d) $\log_b 3 + \log_b p + \log_b q$

**The Power Rule**

In math, we like to take any new problem and if possible, rewrite it so that it looks like a problem we know how to solve. Using this idea, we can rewrite

$$\ln x^3 = \ln(x \cdot x \cdot x)$$

Then we use the product rule to rewrite the right-hand side as

$$\ln x^3 = \ln x + \ln x + \ln x$$

and then we can use algebra to collect the like terms on the right-hand side to get

$$\ln x^3 = 3\ln x\ .$$

Generalizing, if we have the logarithm of a number raised to a power, we can apply the power rule:

$$\log_a M^N = N \log_a M\ .$$

***Example***

Use the power rule to write an equivalent expression for the following:

a) $\log y^{10}$

b) $\ln 2^x$

c) $\log_3 5^7$

d) $\log \sqrt{y}$

e) $\log_x p^q$

f) $\ln m^{-1}$

Answers:

a) $10\log y$

b) $x\ln 2$

c) $7\log_3 5$

d) $\frac{1}{2}\log y$

e) $q\log_x p$

f) $-\ln m$

**The Quotient Rule**

Once again, we will try to use our previous ideas to develop a property of logarithms. Consider the logarithm of a quotient. We can try to rewrite the quotient to be a product instead, since we know the product rule.

$$\log_a\left(\frac{M}{N}\right)=\log_a\left(M\cdot N^{-1}\right)$$

We then use the product rule to expand the right-hand side.

$$\log_a\left(\frac{M}{N}\right)=\log_a M+\log_a\left(N^{-1}\right)$$

But then the last term can be simplified using the power rule to give the quotient rule:

$$\log_a\left(\frac{M}{N}\right)=\log_a M-\log_a N\ .$$

***Example***

Use the quotient rule to write an equivalent expression for the following:

a) $\log\frac{x}{4}$

b) $\ln\frac{a}{b}$

c) $\ln\frac{x}{yz}$

d) $\log_b \frac{2m}{n}$

Answers:

a) $\log x - \log 4$

b) $\ln a - \ln b$

c) $\ln x - \ln y - \ln z$

d) $\log_b 2 + \log_b m - \log_b n$

**Simplification**

Remember that, according to our original definition of logarithm, that $\log_a b$ means "what do I have to raise *a* to the power of to get *b*?" Therefore,

$$\log_a a^x = x$$

Taking this idea a step further, then for any base *a*, $a = a^1$ so that $\log_a a = \log_a a^1 = 1$, and generally

$$\log_a a = 1.$$

Similarly, $\log_a 1 = \log_a a^0 = 0$, so

$$\log_a 1 = 0$$

for any base *a*.

***Example***

Simplify:

a) $\log_x x^4$

b) $\log_a \sqrt[3]{a}$

c) $\ln e^x$

d) $\log 10^3$

e) $\ln \sqrt[5]{e}$

f) $\log_b \frac{1}{b}$

Answers:

a) 4

b) 1/3

c) $x$

d) 3

e) 1/5

f) –1

***Example***

Write each expression as a single logarithm and simplify:

a) $\log_b x^2 + \log_b x^{-2}$

b) $5\log x - 2\log y$

c) $\ln 8 - \ln 2$

d) $\log 2 + \log 5$

e) $\frac{1}{2}\log 400 - \log 2$

Answers:

a) 0

b) $\frac{x^5}{y^2}$

c) $\ln 4$

d) 1

e) 1

***Example***

Write each expression in terms of ln 3 and/or ln $x$:

a) $\ln\frac{1}{3}$

b) $\ln\sqrt{3}$

c) $\ln 27x$

d) $\ln(3x)^5$

Answers:

a) –ln 3

b) ½ ln 3

c) 3 ln 3 + ln $x$

d) 5 ln 3 + 5 ln $x$ or 5(ln 3 + ln $x$)

***Example***

Given that $\log_a x = 2$ and $\log_a y = 3$, evaluate:

a) $\log_a xy$

b) $\log_a \frac{y}{x}$

c) $\frac{\log_a y}{\log_a x}$

d) $\log_a \sqrt{x}$

e) $\log_a x^3$

Answers:

a) 5

b) 1

c) 3/2

d) 1

e) 6

**The Base-Change Formula**

Up until now, we've only been able to calculate decimal equivalents for logarithms with base 10 or *e*, since those are the only two bases available on our calculators. Now let's learn a method to calculate the decimal approximation for logarithms of any base. Suppose we had the equation

$$a^x = M$$

and tried to solve it in two different ways. Our first way might be to rewrite the equation into the equivalent logarithmic form by bringing the base *a* down to get

$$x = \log_a M \text{ .}$$

But we could also try taking the logarithm base 10 of both sides of the original equation, like so:

$$a^x = M$$
$$\log a^x = \log M$$

Then apply the power rule to get

$$x \log a = \log M$$

and solve for *x*:

$$x = \frac{\log M}{\log a}$$

As these two methods are equivalent, the two answers we get are equivalent and

$$\log_a M = \frac{\log M}{\log a} \text{ .}$$

More generally, for any positive number M and any bases *a* and *b*, the base-change formula is:

$$\log_a M = \frac{\log_b M}{\log_b a} \quad \left( = \frac{\log M}{\log a} = \frac{\ln M}{\ln a} \right)$$

***Example***

Use the base-change formula to calculate each logarithm below to four decimal places:

a) $\log_3 5$

b) $\log_{12} 0.3$

c) $\log_{0.2} 9000$

d) $\log_{0.1} 0.3$

Answers:

a) $\log_3 5 = \frac{\log 5}{\log 3} = 1.4650$

b) $\log_{12} 0.3 = \frac{\log 0.3}{\log 12} = -0.4845$

c) $\log_{0.2} 9000 = \frac{\log 9000}{\log 0.2} = -5.6572$

d) $\log_{0.1} 0.3 = \frac{\log 0.3}{\log 0.1} = 0.5229$

Note that you can check your answers: if you take the last example and calculate $0.1^{0.5229}$, you get 0.299985, which is almost equal to 0.3. The reason it's not **exactly** equal is because 0.5229 has been rounded off to four decimal places. If you check by using the full number in your calculator, 0.52287874528033, you should get a better approximation, and your calculator will round that answer to exactly 0.3. Check!