# Section 5.4: Solving Equations

In this section, we'll look at solving both exponential and logarithmic equations.

**Solving Exponential Equations**

Recall that an exponential function is a function in which the variable is in the exponent. Similarly, exponential equations have the variable in the exponent as well. Examples are

$$2^{x+1} = 64 \text{ and } 3^x = 20 \text{ .}$$

Now, we can solve the first one using the one-to-one property of exponential functions, which says that if the bases match, then so do the exponents. Rewriting, we find that

$$\begin{aligned} &2^{x+1} = 64 \\ &2^{x+1} = 2^6 \\ &x + 1 = 6 \\ &x = 5 \end{aligned}$$

Now, let's look at the second one. We cannot use the previous procedure here because we can't match the bases (that would require writing 20 as some power of 3, which we don't know how to do). Therefore, we have to use logarithms.

$$\begin{aligned} &3^x = 20 \\ &x = \log_3 20 \end{aligned}$$

Now, the solution $\log_3 20$ is an exact answer. We could use the base-change formula to get it into a form that we can approximate on our calculator, like so:

$$x = \frac{\log 20}{\log 3} \approx 2.72683$$

Notice, though, that we could have come up with the same solution using a different method. Instead of changing the exponential equation to a logarithmic equation of the same base, what would happen if we took the logarithm, base 10, of both sides:

$$\begin{aligned} &3^x = 20 \\ &\log 3^x = \log 20 \end{aligned}$$

Using the power rule on the left-hand side, we get

$$x\log 3 = \log 20$$
$$x = \frac{\log 20}{\log 3}$$

which you'll notice is the same solution as the one we got after using the base-change formula. Then if we were to plug this into our calculators, we should get the same decimal approximation as before.

But what if we have an expression in the exponent instead of just *x*? How do we solve $3^{x-5} = 20$?

Well, we apply the same procedure, and then use the properties of equality for solving equations to isolate our variable *x*:

$$3^{x-5} = 20$$
$$x - 5 = \log_3 20$$
$$x = \log_3 20 + 5$$

and if we wished, we could use the base-change formula to get the decimal approximation

$$x = \frac{\log 20}{\log 3} + 5 \approx 5.72683$$

What about $3^{2x-1} = 20$?

$$3^{2x-1} = 20$$
$$2x - 1 = \log_3 20$$
$$2x = \log_3 20 + 1$$
$$x = \frac{\log_3 20 + 1}{2}$$

and once again we can use the base-change formula to get a decimal approximation if we wished.

***Example***

Solve $e^{7-x} = 3$.

Answer:

$$e^{7-x} = 3$$
$$7 - x = \ln 3$$
$$-x = \ln 3 - 7$$
$$x = -\ln 3 + 7$$
$$x = 7 - \ln 3$$
$$x \approx 5.90139$$

So 7 – ln 3 is the exact answer and 5.9 is a decimal approximation to one decimal place.

***Example***

Solve $5^{2x+6} = 152$.

Answer:

$$5^{2x+6} = 152$$
$$\log 5^{2x+6} = \log 152$$
$$(2x+6)\log 5 = \log 152$$
$$2x + 6 = \frac{\log 152}{\log 5}$$
$$2x = \frac{\log 152}{\log 5} - 6$$
$$x = \frac{1}{2}\left(\frac{\log 152}{\log 5} - 6\right)$$
$$x \approx -1.43924$$

Note also that you can check your answer by substituting *x* back into the original equation: $5^{2(-1.43924)+6}$, which your calculator should tell you is equal to 152.

**Solving Exponential Equations Involving Coefficients**

What if our equation has a coefficient in front of the log, as in our example below?

$$2e^{x+1} = 24$$

The easiest way to handle this is to divide both sides by 2 in order to reduce it to an equation that we know how to solve:

$$2e^{x+1} = 24$$
$$e^{x+1} = 12$$
$$x + 1 = \ln 12$$
$$x = \ln 12 - 1$$
$$x \approx 1.48491$$

A common mistake you might be tempted to make is to try to change to the logarithmic form of the equation before dividing by two. This is tempting, but in order to do it properly, you need to use your properties of logarithms correctly:

$$\ln\left(2e^{x+1}\right) = \ln 24$$
$$\ln 2 + \ln e^{x+1} = \ln 24$$
$$\ln e^{x+1} = \ln 24 - \ln 2$$
$$x + 1 = \ln 24 - \ln 2$$
$$x = \ln 24 - \ln 2 - 1$$

You might look at this and say that the solution is different from the previous method's answer, but remember the quotient rule: $\ln 24 - \ln 2 = \ln \frac{24}{2} = \ln 12$. So the second method gives exactly the same result as the first method. However, it's a bit trickier than the first because you need more properties of logs, so the first method would be the recommended one. In other words, if you can, then divide both sides by the coefficient on the exponential term to get the "bare exponential" on one side, and then change to the equivalent logarithmic equation.

**Solving Logarithmic Equations**

A logarithmic equation is (you've guessed it!) an equation with a logarithm in it. Our general method of solution will be to find out where the variable is. If the variable is buried within the logarithm, such as the example

$$\log_5\left(x+3\right) = 2$$

then our procedure will involve rewriting the equation into exponential form. For our example, this will be our next step:

$$\log_5\left(x+3\right) = 2$$
$$x + 3 = 5^2$$

and you can now see that this is a single equation in one variable and we will use our usual equation-solving steps to isolate *x*.

$$x+3=25$$
$$x=22$$

As our solution is a nice integer, we don't need to worry about decimal approximations for our answer.

What if there are logarithms on both sides of the equation? There is a property of logarithms similar to the one-to-one property of exponential functions. Namely, if $\log_a m = \log_a n$, then $m = n$ provided that both logarithms have the same base. Let's move to an example:

$$\log_5(2x-3) = \log_5(x+1)$$
$$2x-3 = x+1$$
$$x=4$$

***Example***

Solve $\log_3 8 = \log_3(1-x)$.

Answer:

$$\log_3 8 = \log_3(1-x)$$
$$8 = 1-x$$
$$x = 1-8$$
$$x = -7$$

Notice that the base is completely irrelevant to the solution, provided that it is the same on both sides.

What if the variable is buried within the base of the exponent, as in the following example?

$$\log_{2x} 36 = 1$$
$$36 = (2x)^1$$
$$36 = 2x$$
$$x = 18$$

***Example:***

Solve $\log_{x-1} 27 = 3$.

Answer:

$$\log_{x-1} 27 = 3$$
$$27 = (x-1)^3$$
$$3 = x - 1$$
$$x = 4$$

Note: if you're wondering how we got from the second step to the third step, we just took the cube root of both sides. The cube root of 27 is 3 and the cube root of anything cubed is just itself. It's a bit more tricky if we look at raising to an even power, but for the purposes of this class, we'll restrict ourselves to the odd powers for ease of use.