Section 5.5: Applications

Now that we have the tools to solve equations in which the variable is in the exponent, we can look at a variety of applications that involve exponential equations. The three types that we will be examining are compound interest, exponential growth, and exponential decay.

Compound Interest Problems

We have seen previously that compound interest may be calculated by

$$A = P\left(1+\frac{r}{n}\right)^{nt},$$

where A is the amount of investment after t years, P is the principal (the original amount of the investment), r is the annual interest rate, n is the number of compounding periods per year, and t is the total time of investment.

If, instead, we look at interest compounded **continuously**, then the equation we are interested in is

$$A = Pe^{rt}$$

where A is the amount of the investment, P is the principal, r is the interest rate, and t is the time of the investment.

Example

Tony decides to put $2000 in a term deposit which is compounded monthly. If his term deposit has an annual interest rate of 2.4%, how long will it take him to double his money?

Answer:

Since Tony is doubling his money, $P = 2000$ and $A = 4000$. Expressed as a decimal, the interest rate r is 0.024. Putting this into the formula, we get

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$
$$4000 = 2000\left(1+\frac{0.024}{12}\right)^{12\times t}$$
$$2 = (1.002)^{12t}$$
$$\ln 2 = 12t\ln 1.002$$
$$t = \frac{\ln 2}{12\ln 1.002} = 28.91$$

Tony will double his money in just under 29 years.

Note that for this example, we didn't actually need to know the original amount. If instead all we knew is that Tony's money had doubled, we would still be able to solve the problem: the initial amount would be *P*, and the final amount *A* would just equal 2*P*. Then when you divide both sides by *P*, the two *P*s would cancel and you'd be left with the same calculation:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$
$$2P = P\left(1+\frac{0.024}{12}\right)^{12\times t}$$
$$2 = (1.002)^{12t}$$
$$\ln 2 = 12t\ln 1.002$$
$$t = \frac{\ln 2}{12\ln 1.002} = 28.91$$

which results in the same answer.

Exponential Growth Problems

Anything that undergoes exponential growth (computer viruses, mosquitoes, fluffy bunnies, the number of computers connected to the Internet) will obey the equation for exponential growth,

$$P = P_o e^{rt}$$

where *P* is the population at time *t*, P_o is the initial population (the population at time *t*=0), *r* is the exponential growth rate, and *t* is the elapsed time. You'll notice that this looks almost identical to the equation for continuous compounding, which is no accident since the two situations behave the same way. You may have seen this equation with *k* instead of *r* for the growth rate and with either *A* and A_o or *N* and N_o for the population variables. These are just conventions and it doesn't matter which version you use – the results should be the same.

For situations involving population growth, there are two ways of indicating how fast a particular population grows. You can either specify the growth rate *r* (expressed as a percentage per year – like an interest rate) or you can give another quantity called the **doubling time**. The doubling time is just the amount of time it takes for a population to grow to twice its original size (in fact, the previous example involving compound interest computed the doubling time for an investment). You can use this quantity to solve exponential growth problems, but it's usually a two-step process: you first have to find the growth rate *r* and then compute the quantity you're actually interested in.

Example

According to Wikipedia, 1.1 billion people currently enjoy regular access to the Web. If the annual growth rate of this population is 6.6%, how long will it take before twice as many people will have access to the Web?

Answer:

Instead of putting the billions of people into the exponential growth equation, I will simplify matters by just saying that $P = 2P_o$. Then

$$P = P_o e^{rt}$$
$$2P_o = Pe^{0.066t}$$
$$2 = e^{0.066t}$$
$$\ln 2 = 0.066t$$
$$t = \frac{\ln 2}{0.066} = 10.5$$

So the doubling time is 10.5 years.

Example

Brazil, on the other hand, has a doubling time of 7.7 years for people having regular access to the Web. If there are 5.6 million people currently connected, how many people will be connected in 10.5 years?

Answer:

Since the problem doesn't explicitly give us *r*, we'll have to calculate it first from the doubling time.

$$P = P_o e^{rt}$$
$$2P_o = P_o e^{r\,7.7}$$
$$2 = e^{7.7r}$$
$$\ln 2 = 7.7r$$
$$r = \frac{\ln 2}{7.7} = 0.090019 = 0.09$$

Brazil's growth rate is 9%. Now we'll put this back into our equation, using a time of 10.5 years. We could either put in 5,600,000 as our original population, or as I've done below, we could just leave our answer in millions of people.

$$P = P_o e^{rt}$$
$$P = 5.6e^{0.09\times 10.5}$$
$$P = 14.4$$

Brazil will have 14.4 million people with regular access to the Web in 10.5 years.

Exponential Decay Problems

There are certain physical phenomena that undergo exponential decay: the rate of cooling or warming of objects at one temperature when moved to surroundings of different temperature, rates of chemical reactions, atmospheric pressure, and the amount of charge on a capacitor, to name a few. For the purposes of this class, we'll concentrate on examples involving radioactive decay. The equation for exponential decay looks very similar to the corresponding equation for exponential growth, with the exception that the exponent is negative:

$$P = P_o e^{-rt}$$

where P is the amount of the radioactive substance at time t, P_o is the initial amount of the substance (the amount at time t=0), r is the exponential decay rate, and t is the elapsed time. You will often see this equation using A and A_o or N and N_o for the amount, but for simplicity I will stick with the P notation.

It's also acceptable to use the original equation, $P = P_o e^{rt}$, for exponential decay. The only difference is that when you solve for r, you'll find that it's a **negative** quantity, so the end result looks just like the equation above containing the negative sign. It does not matter which convention you use, but you have to stick with the same convention throughout any one problem.

When a radioactive substance decays, how fast it is decaying is usually described by the **half-life**, the time it takes for only half of the original substance to be left. Once this is given, then the decay rate r may be calculated as before as an intermediate step in solving the problem.

Example

Carbon-14 has a half-life of 5730 years. If there was originally 15.0 grams of carbon-14, how much will be left after 1000 years? How much will have decayed away?

Answer:

First, we'll need to calculate r for $P = \frac{1}{2} P_o$:

$$P = P_o e^{-rt}$$
$$\frac{1}{2} P_o = P_o e^{-r5730}$$
$$\frac{1}{2} = e^{-5730r}$$
$$\ln \frac{1}{2} = -5730r$$
$$r = -\frac{\ln(1/2)}{5730} = 0.000121$$

Now that we've found r, we'll find P after 1000 years:

$$P = P_o e^{-rt}$$
$$P = 15.0 e^{-0.000121 \times 1000}$$
$$P = 13.3$$

There will be 13.3 grams of carbon-14 left.

To find the amount decayed away, subtract the amount left from the initial amount, so (15.0 – 13.3) grams, or 1.7 grams will have decayed away.