Section 1.10: More Laws of Logic

De Morgan's Laws

Let's examine the following truth table.

p	q	$p \wedge q$	$\overline{p \wedge q}$	$\overline{p}$	$\overline{q}$	$\overline{p} \vee \overline{q}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

From this table, we can see that

$$\overline{p \wedge q} \Leftrightarrow \overline{p} \vee \overline{q} \ .$$

We can draw a similar table to show that

$$\overline{p \vee q} \Leftrightarrow \overline{p} \wedge \overline{q} \ .$$

These are called **De Morgan's laws**. Notice that if we replaced all of the logic symbols in the table with the equivalent set symbols and also by Boolean algebra notation, we'd get

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \qquad \overline{AB} = \overline{A} + \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A + B} = \overline{A}\ \overline{B}$$

The set versions should look familiar, since we previously proved that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ using Venn Diagrams in Section 1.4.

Distributive Laws

Again, let's examine the following truth table.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0

1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

From this table, we can see that

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

We can draw a similar table to show that

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r) \text{ .}$$

These are called the **distributive** laws. Notice that if we replaced all of the logic symbols in the table with the equivalent set symbols and also by Boolean algebra notation, we'd get

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad A(B+C) = AB + AC$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad A + BC = (A+B)(A+C)$$

Absorption Laws

Again, let's examine the following truth table.

p	q	$p \vee q$	$p \wedge (p \vee q)$	$\overline{p}$	$\overline{p} \vee q$	$p \wedge (\overline{p} \vee q)$
0	0	0	0	1	1	0
0	1	1	0	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	1	1

From this table, we can see that

$$p \wedge (p \vee q) \Leftrightarrow p$$
$$p \wedge (\overline{p} \vee q) \Leftrightarrow p \wedge q$$

We can draw a similar table to show that

$$p \vee (p \wedge q) \Leftrightarrow p$$
$$p \vee (\overline{p} \wedge q) \Leftrightarrow p \vee q$$

These are called the **absorption** laws. Notice that if we replaced all of the logic symbols in the table with the equivalent set symbols and also by Boolean algebra notation, we'd get

$A \cap (A \cup B) = A$	$A(A+B) = A$
$A \cap (\overline{A} \cup B) = A \cap B$	$A(\overline{A} + B) = AB$
$A \cup (A \cap B) = A$	$A + AB = A$
$A \cup (\overline{A} \cap B) = A \cup B$	$A + \overline{A}B = A + B$

Simplifying Logical Expressions

The real power of these laws lies in simplifying logical expressions and in proofs. (Remember – one law per line, must write name of law!)

Example

Simplify $(p \wedge q) \vee (p \wedge \overline{q})$.

Answer:

$(p \wedge q) \vee (p \wedge \overline{q})$

$p \wedge (q \vee \overline{q})$	distributive
$p \wedge 1$	complement
p	identity

Example

Simplify $AB(\overline{A} + \overline{B})$.

Answer:

$AB(\overline{A} + \overline{B})$

$AB\overline{A} + AB\overline{B}$	distributive
$A\overline{A}B + AB\overline{B}$	commutative

$0 \cdot B + A \cdot 0$ complement

$0 + 0$ identity

0 definition of "or"

Note that for many of these exercises, there is more than one way to answer. Another equally valid simplification looks like the following.

$AB(\bar{A} + \bar{B})$

$AB \; \overline{AB}$ De Morgan's

0 complement

This is a much shorter answer, but does require a flash of insight at the $(\bar{A} + \bar{B})$ pattern.

Example

Show that $\overline{A + \bar{A}\,\bar{B}} = \bar{A}\,B$.

Answer:

Let's examine the left-hand side.

$\overline{A + \bar{A}\,\bar{B}}$

$\overline{A + \bar{B}}$ absorption

$\bar{A}\;\bar{\bar{B}}$ De Morgan's

$\bar{A}\;B$ complement

and the fact that the left-hand side is equivalent to $\bar{A}\,B$ completes our proof.

Example

Show that $(A + \bar{C})(\bar{C} + AB) = AB + \bar{C}$.

Answer:

Let's examine the left-hand side.

$(A + \overline{C})(\overline{C} + AB)$	
$(\overline{C} + A)(\overline{C} + AB)$	commutative
$\overline{C} + A(AB)$	distributive
$\overline{C} + (AA)B$	associative
$\overline{C} + AB$	idempotent
$AB + \overline{C}$	commutative

QED. (QED is short for the Latin phrase "quod erat demonstrandum", which means "it has been demonstrated.)