Section 1.11: The Conditional

The Conditional Connective

Suppose we have two propositions, p and q. Remembering that connectives are operations which join two or more propositions (like "and" and "or"), the conditional connective is

If p, then q.

In symbols, this is written as $p \rightarrow q$, and when read aloud, you say "p implies q". When using the conditional, the first proposition is called the **hypothesis** and the second is called the **conclusion**.

There are other ways to state the conditional. "If p, then q" is equivalent to

a) p implies q
b) q, if p
c) p is sufficient for q
d) q is necessary for p
e) p only if q

We'll only be using the "If p, then q" and "p implies q" conventions in this course.

But what does the conditional mean? (My thanks go to Gilles Cazelais for providing the idea for the following example.) Suppose you have an insurance contract which reads:

If your house burns down, then the insurance company give you $1 000 000.

Suppose your house burns down. Under the contract, the insurance company must give you one million dollars. If it doesn't, the contract has been violated. But if your house doesn't burn down and the company doesn't give you any money, the contract still holds. If your house doesn't burn down and out of the boundless generosity the company give you one million dollars anyway, the contract still holds. The only circumstances under which the contract is violated is when your house does burn down but the company fails to give you one million dollars. This leads to the following truth table.

House burns down?	You get $1 000 000?	The contract holds.
no	no	yes
no	yes	yes
yes	no	no
yes	yes	yes

To generalize to the propositions p and q,

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

What does this mean? It means that if p is true and q is false, then the implication $p \rightarrow q$ cannot also be true. It also means that if $p \rightarrow q$ is true, then you cannot have p true and q false at the same time.

Let's look at another example. Suppose that the following conditional is true: "If Barney is a dog, then Barney has four legs." This means that if the first proposition, "Barney is a dog," is true, then only one conclusion may be reached, that the second is true and Barney has four legs. However, if p is false and Barney is not a dog, then our conditional doesn't have anything to say about the number of legs Barney may have. If Barney is not a dog (Barney is a snake, octopus, bug, person, or pond), then Barney may not have four legs. If Barney is not a dog (Barney is a cat, giraffe, woolly mammoth, or table), Barney may have four legs. Our conditional does not deal with what you may conclude when p is false.

Example:

Suppose that the statement "If Pat sleeps in, she will be late for class" is true. Answer the following questions.

a) Pat sleeps in. Is she late for class?
b) Pat does not sleep in. Is she late for class?
c) Pat is late for class. Did she sleep in?
d) Pat is not late for class. Did she sleep in?

Answer: a) Yes. b) Maybe. (Perhaps she ran into traffic or was eaten by bears. Remember that the conditional has nothing to say when the first proposition is false.) c) Maybe. (Again, maybe there was another reason for her lateness.) d) No.

The converse

If $p \rightarrow q$ is true, is it also true that $q \rightarrow p$? If $p \rightarrow q$ is called the conditional, then $q \rightarrow p$ is called the **converse**. Let's use our previous example again, which was "If Barney is a dog, then Barney has four legs." If this statement is true, is it also true that "If Barney has four legs, then Barney is a dog"?

Clearly this second statement is not also true, since Barney could be a four-legged creature that is not a dog, such as a cat, mouse, grizzly bear, or mountain goat.

Let's write out the truth table for the conditional and the converse.

p	q	p→q	q→p
F	F	T	T
F	T	T	F
T	F	F	T
T	T	T	T

Here's how to fill in the columns for any logical expression containing →. Let's call the propositions "first" and "second" so we don't get confused with p and q. For the conditional first→second, it will be true for all cases **except** when the first is true and the second is false. So for q→p, look for the row in which q is true (rows 2 and 4) and p is false (1 and 2). Then q is true and p is false only for row 2. Therefore, all rows except for the second get True and the second row gets False. So you can also see from the truth table that the conditional p→q and the converse q→p are not logically equivalent.

The contrapositive

What about the contrapositive, $\bar{q} \rightarrow \bar{p}$? For our familiar example, that would be asking whether "If Barney is a dog, then Barney has four legs" is equivalent to "If Barney does not have four legs, then Barney is not a dog". This at least looks a little more promising. Let's try the truth table, but this time using 1s and 0s instead of True and False.

p	q	$\bar{p}$	$\bar{q}$	p→q	$\bar{q} \rightarrow \bar{p}$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	0	0	1	1

Remember, you'll fill in all 1s except for the row where the first is true and the second is false. So look for the rows with $\bar{q}$ true (rows 1 and 3) and $\bar{p}$ false (rows 3 and 4). So the row with the zero will be the third row. As the last two columns are identical, then the conditional p→q is equivalent to the contrapositive $\bar{q} \rightarrow \bar{p}$.

Example

Write the contrapositive for the statement, "If today is sunny, Pat will work in the garden."

Answer: To get the contrapositive, negate each proposition and then reverse the order. So the contrapositive is "If Pat is not working in the garden, then today is not sunny."

The Inverse

If p→q is the conditional, then the proposition $\overline{p} \rightarrow \overline{q}$ is called the inverse. So if the conditional is "If Barney is a dog, then Barney has four legs", then the inverse of that would be "If Barney is not a dog, then Barney does not have four legs." You can see directly from this example that the conditional and the inverse are not equivalent!

Here's what the truth table looks like. (Remember that the → means that the value is 1 except when the first is true and the second is false.)

p	q	$\overline{p}$	$\overline{q}$	p→q	$\overline{p} \rightarrow \overline{q}$
0	0	1	1	1	1
0	1	1	0	1	0
1	0	0	1	0	1
1	1	0	0	1	1

Example:

Draw the truth tables for the conditional (p→q), the converse (q→p), the inverse ($\overline{p} \rightarrow \overline{q}$), and the contrapositive ($\overline{q} \rightarrow \overline{p}$). Are any of these propositions logically equivalent?

Answer: Here's the big truth table.

p	q	$\overline{p}$	$\overline{q}$	p→q	q→p	$\overline{p} \rightarrow \overline{q}$	$\overline{q} \rightarrow \overline{p}$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

Since the 5^{th} and 8^{th} columns are identical, the conditional (p→q) is logically equivalent to the contrapositive ($\overline{q} \rightarrow \overline{p}$). Since the 6^{th} and 7^{th} columns are identical, the converse (q→p) is logically equivalent to the inverse ($\overline{p} \rightarrow \overline{q}$).

The "or" form of the conditional

Can the conditional p→q be rewritten using our basic connectives "and", "or", and "not"? Yes, it can, because you can see by the truth table below that p→q is logically equivalent to $\overline{p} \vee q$ (not-p or q).

p	q	$\bar{p}$	$\bar{p} \vee q$	$p \to q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

This means that "If Barney is a dog, then Barney has four legs" is logically equivalent to "Either Barney is not a dog or he has four legs."

Example

Consider the conditional $p \to q$. Is the converse $(q \to p)$ logically equivalent to $\bar{p} \vee q$, $\bar{p} \wedge q$, $p \vee \bar{q}$, or $p \wedge \bar{q}$?

Answer: Let's write in the truth table and compare columns.

p	q	$\bar{p}$	$\bar{q}$	$q \to p$	$\bar{p} \vee q$	$\bar{p} \wedge q$	$p \vee \bar{q}$	$p \wedge \bar{q}$
0	0	1	1	1	1	0	1	0
0	1	1	0	0	1	1	0	0
1	0	0	1	1	0	0	1	1
1	1	0	0	1	1	0	1	0

As can be seen from the table, the columns for $q \to p$ and $p \vee \bar{q}$ are identical, so these two expressions are logically equivalent.

De Morgan's Law and the Contrapositive

Consider the conditional $(p \wedge q) \to r$. The contrapositive would be $\bar{r} \to \overline{p \wedge q}$. Applying De Morgan's Law gives $\bar{r} \to \bar{p} \vee \bar{q}$. Notice the change from the "and" in the conditional to the "or" in the modified contrapositive. Forgetting to make that change is an easy trap to fall into.

Example

Consider the conditional "If Pat sleeps in or runs into traffic, she will be late for class." What is the contrapositive? Use De Morgan's Law to find your answer.

Answer: The contrapositive is "If Pat is not late for class, then she didn't sleep in **and** did not run into traffic." The "or" changes into an "and" because of De Morgan's Law.