Section 1.3: Operations on Sets

Union of Sets

Suppose we wished to combine sets A and B. The set of all elements that belong either to set A or set B or both is called the "union of A and B" and is written as $A \cup B$. The formal definition is then

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

This mathematical sentence can be read as "the union of A and B is equal to all *x* such that *x* belongs to A **or** *x* belongs to B."

Example:

Let A = {1, 2, 3, 4} and B = {1, 3, 5, 7, 9}. Find $A \cup B$.

Answer: $A \cup B$ = {1, 2, 3, 4, 5, 7, 9}. Since repetition doesn't matter, I don't bother to write either the 1 or the 3 twice.

Example:

Let C = {1, 3, 5, …} and D = {2, 4, 6, …}. Find $C \cup D$.

Answer: $C \cup D$ = {1, 2, 3, 4, 5, 6., …} = {1, 2, 3, …}. If you are feeling lazy, it would also be perfectly correct to just write that $C \cup D$ = N.

Intersection of Sets

Another way to combine sets A and B would be to consider the set of all elements common to both A and B. This is called the "intersection of A and B" and is written as $A \cap B$. The formal definition is then

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

This mathematical sentence can be read as "the union of A and B is equal to all *x* such that *x* belongs to A **and** *x* belongs to B."

Example:

Let A = {1, 2, 3, 4} and B = {1, 3, 5, 7, 9}. Find $A \cap B$.

Answer: $A \cap B$ = {1, 3} since only 1 and 3 are in both A and B.

Example:

Let C = {1, 3, 5, …} and D = {2, 4, 6, …}. Find $C \cap D$.

Answer: Since there are no elements that are in both C and D, $C \cap D = \varnothing$.

Remember that $\cup$ stands for "either" or "or", while $\cap$ stands for "both" or "and." Unfortunately, the use of the words "union" and "and" in English are sometimes interchangeable, while in logic and set theory, "union" really means "or". It's easy to get confused, so practice is key!

Example

For any set A, what is $A \cup \varnothing$? $A \cap \varnothing$?

Answer: Since $\varnothing$ is empty, joining it to A doesn't add any extra elements and $A \cup \varnothing = A$. Similarly, $A \cap \varnothing = \varnothing$, since the two sets do not have any elements in common.

Example

If $A \subseteq B$, what is $A \cup B$? $A \cap B$?

Answer: Since every element of A is also in B, A doesn't contribute any extra elements and $A \cup B = B$. Similarly, $A \cap B$ is the set of elements in common between the two sets, which is just A. So $A \cap B = A$ for all sets in which A is a subset of B.

Similarly, for all sets in which $A \subset B$, $A \cup B = B$ and $A \cap B = A$.

Combinations of Unions and Intersections

What if we wish to combine two or more operations? Suppose you were asked to find $(A \cup B) \cap C$? The nice thing is that the use of brackets here means exactly what you would expect, which is to evaluate $A \cup B$ first, and then intersect the result with C.

Example:

Let A = {1, 2, 3}, B = {2, 4, 6}, and C = {1, 3, 5}. Find $(A \cup B) \cap C$ and $A \cup (B \cap C)$.

Answer: To find $(A \cup B) \cap C$, first find $A \cup B$ = {1, 2, 3, 4, 6}. The intersection of this new set with C gives {1, 3}.

To find $A \cup (B \cap C)$, find $B \cap C = \varnothing$. Then $A \cup (B \cap C) = A \cup \varnothing = A$.

Notice that although $(A \cup B) \cap C$ and $A \cup (B \cap C)$ contain the same sets and operations, the order in which you do the $\cup$ or $\cap$ is important, giving different results depending on where the brackets go. Later in this course we will learn which operation to evaluate first if brackets aren't present. For the time being, I will state which operation I want done first by using brackets.

Negation of Sets (Complement)

If A = {1, 2, 3}, what would be the negative of this set, called "not A"? If you think about it, it would have to depend on what pool of numbers A is drawn from. For example, if A were drawn from the natural numbers, then the negation of A would be {4, 5, 6,…}. If A were drawn from the integers, then the negation of A would be all integers except for 1, 2, and 3. This brings us to the idea of the universal set U. If we are talking about the negation of a set, we need to specify the pool of numbers, the universal set, that this set is taken from.

The negation of A is usually written as $\overline{A}$, and commonly just called "not-A".

> ***Example:***
>
> Let A = {1, 2, 3} and U = {0, 1, 2, … 6}. Find $\overline{A}$.
>
> Answer: $\overline{A}$ will be all members of U that are not in A, so $\overline{A}$ ={0, 4, 5, 6}.

There are some nice properties that you get when you combine A and $\overline{A}$. Notice that $A \cup \overline{A}$ = {1, 2, 3}$\cup${0, 4, 5, 6} = U. I hope this makes a certain amount of sense: the set of all those in A and all those not in A would be U. Similarly, $A \cap \overline{A}$ = $\{1,2,3\} \cap \{0,4,5,6\}$ = $\varnothing$, since there should be no overlap between "all those in A" and "all those not in A". These two properties, $A \cup \overline{A} = U$ and $A \cap \overline{A} = \varnothing$, are true for all sets A and not just this example.

> ***Example:***
>
> Let B = {y | y = a negative integer} and U = Z. Find $\overline{B}$, $B \cup U$, and $B \cap U$.
>
> Answer: Note that B = {…, –3, –2, –1} and Z = $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
> So that makes $\overline{B}$ = {0, 1, 2, 3, …}.
>
> $B \cup U$ = {…, –3, –2, –1, 0, 1, 2, 3, …} = U
>
> $B \cap U$ = {…, –3, –2, –1} = B

If we look at the previous example, we notice that $B \cup U$ = U and $B \cap U$ = B. These two relationships hold true for all B taken from U and not just this particular example. The reason for this is that if you consider that all members of B are taken from the set U, then $B \subseteq U$, and we've already discussed how the union of a set and its subset is just the original set, as there are no members of the subset not in the original set.

Example:

Let C = {2, 4, 6, …} and U = { x | x is an even positive integer}. Find $\overline{C}$, $C \cup U$, and $\overline{C} \cap U$.

Answer: Notice that C = U. Therefore, there are no elements of U that are not in C, so $\overline{C} = \varnothing$.

$C \cup U = U$ no matter what C is. However, for this special case, you could also write $C \cup U = C$ as well.

$\overline{C} \cap U = \overline{C}$ in general. For this special case, it's also true that $\overline{C} \cap U = \varnothing$.

Example:

Let A = {1, 2, 3, 4}, B = {1, 3, 5, 7}, and U = {1, 2, 3, … 8}. Find $\overline{A}$, $A \cup \overline{B}$, and $\overline{A \cup B}$.

Answer: $\overline{A}$ = {5, 6, 7, 8}

To find $A \cup \overline{B}$, we first find that $\overline{B}$ = {2, 4, 6, 8}. Then $A \cup \overline{B}$ = $\{1, 2, 3, 4, 6, 8\}$.

To find $\overline{A \cup B}$, we should first find $A \cup B$ and then negate the result. $A \cup B$ = {1, 2, 3, 4, 5, 7}, so then $\overline{A \cup B}$ = {6, 8}.

Let's finish this section with a particularly evil example.

Example:

Let A = {1, 2, 3, 4}, B = {1, 3, 5, 7}, and U = {1, 2, 3, … 8}, as before. Are the following statements true or false?

$A \in N$

$3 \in (1, 2, 3)$

$2 \subseteq U$

Answer: They are all false. First, it's true that A is a subset of N, but it's not an **element** of N (N is a set of numbers, not a set of sets).

Second, (1,2,3) is not a set, since it uses round brackets and not set braces, {}. So although it is true that $3 \in \{1, 2, 3\}$, it cannot be a member of the non-set (1, 2, 3) – which is actually an ordered triple. We'll learn more about ordered triples when we study relational algebra.

Third, although 2 does belong to the set U, 2 cannot be a **subset** of U. In order to be a subset, the object has to first be a set! So you could say that $2 \in U$ or $\{2\} \subseteq U$ but $2 \not\subseteq U$.

And yes, all three of these questions are testing whether you recognize incorrect notation!

Python and Operations on Sets

In Python, there are two ways of specifying unions and intersections, as shown below.

```
>>> A={1,2,3}
>>> B={1,3,5}
>>> A.union(B)
set([1, 2, 3, 5])
>>> A|B
set([1, 2, 3, 5])
>>> A.intersection(B)
set([1, 3])
>>> A&B
set([1, 3])
>>> 
```

Figure 1: Unions and Intersections of Sets in Python