Section 1.5: Introduction to Logic

Propositions

In logic, a proposition is a statement that is either true or false but not both. The statement must also be unambiguous.

Examples of statements that **are** propositions:

Jon Stewart is the host of the Daily Show on the Comedy Network.

Lego Star Wars is a video game.

The number π is exactly equal to 3.

A proposition can clearly be false, as in the last statement, while still being a proposition.

Examples of statements that are **not** propositions:

Will you do your homework tonight?

Please pass the butter.

She was late for class this morning.

The first is not a proposition because questions cannot be propositions. (Note that the answer to the question may very well be a proposition.) The second one is a command and cannot be said to be either true or false. The third of these examples is not a proposition because, taking the statement on its own, the truth value depends on who "she" is. If, however, the statement was expanded to become, "My roommate's name is Laura and she was late for class this morning," then the "she" is clearly defined to be Laura and the whole sentence is a proposition.

Taking this idea one step further, we can consider the "she" in the third example to behave like a variable, and whether the full statement "she was late" is true or false must depend on what the value of the variable "she" is. Similarly, in programming it is very common to evaluate the value (true/false) of propositions like "$x = 3$" or "$y < 5$" in statements like:

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if x = 3 then print "Hello World"
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provided that, like she/Laura, the value of *x* has previously been defined.

Since writing propositions out using English sentences is unwieldy, we frequently use variables to denote propositions. In symbolic logic, we usually use the letters p, q, r, s, t, etc., for propositions. Each of these variables can then have one of two values, true or false. For example, let p = "Lego Star Wars is a video game" and q = "The number π is exactly equal to 3." In this instance, the proposition p is true, since there is a video game called Lego Star Wars and the proposition q is false, since π is the irrational number 3.1415926… which does not repeat and does not terminate.

Operators

"not"

The negation of any proposition p is called (not-p) and is written as $\overline{p}$. You may also see it written using a tilde, ~p, or using this strange symbol, $\neg p$. We will use the $\overline{p}$ notation because it's using the same convention as we will be using in sets and Boolean algebra, so will be easier to remember.

Note that you should be a little careful when negating sentences. For example, the negative of "Pat is happy" is not "Pat is unhappy". There are many other emotions that Pat could have (anger, fear, boredom, etc.). If the first statement is false, then its negation must be true, so between the two you need to cover all possible situations that could arise. It would be safe to say that the negation of "Pat is happy" is that "Pat is not happy", though.

Example:

Are these two sentences the negative of each other?

"The number of students in Math 161 is even."
"The number of students in Math 161 is odd."

Answer: Yes, these two are negatives of each other. Since we never have fractions of students in class, the number of students must be either zero or a natural number (in other words, a whole number). Since whole numbers and natural numbers are either even or odd, these two sentences cover all bases and are negatives of each other. (Yes, zero is an even number.)

Example:

Are these two sentences the negative of each other?

"Pat's Visa account balance is positive."
"Pat's Visa account balance is negative."

Answer: No, these two statements are not negatives of each other. There is a third possible case, "Pat's Visa account balance is zero." So the two statements above don't cover all options. However, if the second statement read "Pat's Visa account balance is negative or zero", then the second statement would be the negation of the first one.

Combining Two or More Propositions Using Connectives

Propositions may be combined using logical operators called connectives. There are three basic connectives that we will study: "and", "or", and "exclusive or". (Oddly enough, the "not" operator is also called a connective, even though it acts on only one entity rather than joining two.)

"and"

If we connect the propositions p and q with "and" (also called the **conjunction**), then "p and q" is true if both p and q are true. The symbol for "and" is ^, so "p and q" is written p ^ q.

> ***Example:***
>
> "Pat does her marking and goes to a movie" is true if and only if Pat both does her marking and goes to a movie. If she does one or the other **but not both**, then the statement "Pat does her marking and goes to a movie" is false. It's also false if she does neither action.

"or"

If we connect the propositions p and q with "or" (also called the **inclusive disjunction**), then "p or q" is true if either p or q or both are true. The symbol for "or" is $\vee$, so "p or q" is written $p \vee q$.

"exclusive or" or "XOR"

If we connect the propositions p and q with "exclusive or" (also called the **exclusive disjunction** and frequently written as XOR), then "p XOR q" is true if either p or q **but not both** are true. The symbol for "exclusive or" is $\oplus$, so "p XOR q" is written $p \oplus q$.

"or" vs. "XOR"

In ordinary English, the word "or" can mean either the "inclusive or" or the "exclusive or", and it is usually up to the reader/listener to decide which one was meant from the context.

> ***Example:***
>
> Which "or" is meant in the following English sentences/phrases?
>
> "Would you like milk or sugar in your tea?"
>
> "Wanted dead or alive"

Answer: For the “milk or sugar” question, the answer could easily be “milk”, “sugar”, “both”, or “neither”. Since “both” is an option, the “inclusive or” is clearly meant.

Since for the “dead or alive” question, the person who is wanted in one of these two states will either be dead or alive but not both, so the “exclusive or” is the best interpretation.

To unambiguously state which “or” is meant in English, the word “or” can be replaced by slightly wordier constructions. The sentence “Would you like milk or sugar or both in your tea?” makes it clear that the “inclusive or” is meant. Replacing “or” by “and/or” has the same result. Using “either … or” or the phrase “but not both” are signals that the “exclusive or” is meant.

In general, if a statement is ambiguous, it is best to seek clarification. If that is not possible, then assuming that “or” means the “inclusive or” is generally the safest bet. For the rest of this course, we will use “or” to mean the “inclusive or”.

Notation

To negate a compound proposition, you put the negation bar over the entire expression. The negation of $p \wedge q$ is then $\overline{p \wedge q}$. The negation bar then behaves in the same say that brackets do. To determine whether the proposition $\overline{p \wedge q}$ is true, you first evaluate $p \wedge q$ and then negate that value.

Order of Operations

To evaluate the proposition $p \vee (q \wedge r)$, you do exactly as you would expect: evaluate $q \wedge r$ and then “or” that value with p. Similarly, $p \vee \overline{q \wedge r}$ requires that you evaluate $q \wedge r$, negate the result, then “or” with p.