Section 1.6: Logical Equivalence

Truth Tables

Let us consider the propositions p and q. Since they are propositions, p is either true or false and q is either true or false. This leads us to four possible combinations of p and q:

1. p is false and q is false
2. p is false and q is true
3. p is true and q is false
4. p is true and q is true

We can combine these possibilities into a table called a truth table. We can add further columns to find out what the value of other compound propositions for each combination of p and q as well. Suppose we wished to find out what the truth table was for $p \wedge q$ and $p \vee q$. Then the table would look like the following.

p	q	$p \wedge q$	$p \vee q$
F	F	F	F
F	T	F	T
T	F	F	T
T	T	T	T

For example, when p is false and q is true (the second row, where p=F and q=T), then $p \wedge q$ is false because one of them is false and $p \vee q$ is true because at least one of them is true.

However, we can also abbreviate the table, changing all Fs to 0s and Ts to 1s. We do this so that there is a good correspondence between these truth tables and the tables we will be learning for sets and Boolean algebra. So another equally correct truth table would be:

p	q	$p \wedge q$	$p \vee q$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

What if we were interested in the truth table for $p \vee \overline{p}$? To get $\overline{p}$, we'll simply negate each value for p in the truth table and then "or" the result as before.

p	$\overline{p}$	$p \vee \overline{p}$
0	1	1
1	0	1

Notice, then, that $p \vee \overline{p}$ is always true. I hope that makes a certain amount of sense: the proposition "Pat's hair is green or Pat's hair is not green" is a statement that is always true.

What if we were interested to find out what happens when we write the truth table for p^1, where 1 means a statement that is always true?

The table would look like:

p	1	p^1
0	1	0
1	1	1

We notice that the last column looks like the first, so p^1 has the same values as p. We say, then, that p^1 is **logically equivalent** to p. We'll talk more about logical equivalence in a bit.

What would the truth table for three propositions look like? We must have eight rows to display all possibilities for p, q, and r. The truth table for $\overline{p}$^(q$\vee\overline{r}$) would then be

p	q	r	$\overline{r}$	$q \vee \overline{r}$	$\overline{p}$	$\overline{p}$^$(q \vee \overline{r})$
0	0	0	1	1	1	1
0	0	1	0	0	1	0
0	1	0	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	1	0	0
1	0	1	0	0	0	0
1	1	0	1	1	0	0
1	1	1	0	1	0	0

It's important to note that the actual order of the rows doesn't matter for the truth table to be complete. However, if you write out the table with the rows in a random order, it's very easy to duplicate one of the previous rows. The duplicate row in and of itself isn't a mistake, but if you stop your table at the correct **total** number of rows, the duplicate means that one of the combinations of your variables is missing, which **is** an error.

Logical Equivalence

Two logical expressions are said to be logically equivalent if they have the same values in their columns in the truth table. We saw in our examples above that $p\vee\overline{p}$ was logically equivalent to 1 and p^1 was logically equivalent to p. The symbol for "logically equivalent to" is ⇔, so $p\vee\overline{p} \Leftrightarrow 1$ and p^1 ⇔ p.

Example:

Is p^(q∨r) logically equivalent to (p^q)∨r?

Answer:

p	q	r	q∨r	p^(q∨r)	p^q	(p^q)∨r
0	0	0	0	0	0	0
0	0	1	1	0	0	1
0	1	0	1	0	0	0
0	1	1	1	0	0	1
1	0	0	0	0	0	0
1	0	1	1	1	0	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

No, these two expressions are not logically equivalent because their columns in the truth table, columns 5 and 7, are not identical. This example shows once more that order of operations is important!

Example

Simplify $(p\wedge q)\vee(\overline{p}\wedge q)$.

Answer:

p	q	$\overline{p}$	p^q	$\overline{p}$^q	$(p\wedge q)\vee(\overline{p}\wedge q)$
0	0	1	0	0	0
0	1	1	0	1	1
1	0	0	0	0	0
1	1	0	1	0	1

Notice that the last column is identical to the column for q. Therefore, $(p \wedge q) \vee (\overline{p} \wedge q)$ is logically equivalent to q, which is the simplified logical expression.

Example

Is $p \oplus q$ logically equivalent to $(p \wedge \overline{q}) \vee (\overline{p} \wedge q)$?

Answer:

p	q	$p \oplus q$	$\overline{p}$	$\overline{q}$	$p \wedge \overline{q}$	$\overline{p} \wedge q$	$(p \wedge q) \vee (\overline{p} \wedge q)$
0	0	0	1	1	0	0	0
0	1	1	1	0	0	1	1
1	0	1	0	1	1	0	1
1	1	0	0	0	0	0	0

Yes, the two expressions are logically equivalent.