Section 1.9: Laws of Logic

You may have noticed some common patterns running through some of the exercises by now. Let's examine those patterns in more detail.

First, let us look at the connections between the three sets of symbols we've used so far.

Logic	p^q	p∨q	$\overline{p}$	F	T
Sets	A∩B	A∪B	$\overline{A}$	∅	U
Boolean Algebra	AB	A+B	$\overline{A}$	0	1

In each case, we have symbols for negation, "or"/union, and "and"/intersection. There are also equivalences with False/True for logic, empty set/universal set for sets, and 0/1 (off/on) for Boolean algebra and logic circuits. Let's see what else they have in common.

Identity Laws

Examining logical symbols first, let's fill in the following truth table.

p	0	1	p^0	p∨0	p^1	p∨1
0	0	1	0	0	0	1
1	0	1	0	1	1	1

From this table, we can see that

$$p \wedge 0 \Leftrightarrow 0$$
$$p \vee 0 \Leftrightarrow p$$
$$p \wedge 1 \Leftrightarrow p$$
$$p \vee 1 \Leftrightarrow 1$$

These are the identity laws, true for any proposition p. Notice that if we replaced all of the logic symbols in the table with the equivalent set symbols and also by Boolean algebra notation, we'd get

$A \cap \emptyset = \emptyset$	$A \cdot 0 = 0$
$A \cup \emptyset = A$	$A+0 = A$
$A \cap U = A$	$A \cdot 1 = A$
$A \cup U = U$	$A+1 = 1$

Idempotent Laws

Similarly, let's examine the following truth table.

p	p^p	p∨p
0	0	0
1	1	1

From this table, we can see that

$$p \wedge p \Leftrightarrow p$$
$$p \vee p \Leftrightarrow p$$

These are called the **idempotent** laws. Notice that if we replaced all of the logic symbols in the table with the equivalent set symbols and also by Boolean algebra notation, we'd get

$$A \cap A = A \qquad A \cdot A = A$$
$$A \cup A = A \qquad A + A = A$$

Complement Laws

Again, let's examine the following truth table.

p	$\overline{p}$	$\overline{\overline{p}}$	$p \wedge \overline{p}$	$p \vee \overline{p}$
0	1	0	0	1
1	0	1	0	1

From this table, we can see that

$$\overline{\overline{p}} \Leftrightarrow p$$
$$p \wedge \overline{p} \Leftrightarrow 0$$
$$p \vee \overline{p} \Leftrightarrow 1$$

These are called the **complement** laws. Notice that if we replaced all of the logic symbols in the table with the equivalent set symbols and also by Boolean algebra notation, we'd get

$$\overline{\overline{A}} = A \qquad \overline{\overline{A}} = A$$
$$A \cap \overline{A} = \varnothing \qquad A \cdot \overline{A} = 0$$
$$A \cup \overline{A} = U \qquad A + \overline{A} = 1$$

Commutative Laws

Again, let's examine the following truth table.

p	q	p^q	q^p	p∨q	q∨p
0	0	0	0	0	0
0	1	0	0	1	1
1	0	0	0	1	1
1	1	1	1	1	1

From this table, we can see that

$$p \wedge q \Leftrightarrow q \wedge p$$
$$p \vee q \Leftrightarrow q \vee p$$

These are called the **commutative** laws. Notice that if we replaced all of the logic symbols in the table with the equivalent set symbols and also by Boolean algebra notation, we'd get

$$A \cap B = B \cap A \qquad AB = BA$$
$$A \cup B = B \cup A \qquad A+B = B+A$$

Associative Laws

Again, let's examine the following truth table.

p	q	r	p^q	(p^q)^r	q^r	p^(q^r)	p∨q	(p∨q)∨r	q∨r	p∨(q∨r)
0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	1	1	1
0	1	0	0	0	0	0	1	1	1	1
0	1	1	0	0	1	0	1	1	1	1
1	0	0	0	0	0	0	1	1	0	1
1	0	1	0	0	0	0	1	1	1	1
1	1	0	1	0	0	0	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1

From this table, we can see that

$$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$$
$$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$$

These are called the **associative** laws. Notice that if we replaced all of the logic symbols in the table with the equivalent set symbols and also by Boolean algebra notation, we'd get

$$(A \cap B) \cap C = A \cap (B \cap C) \qquad (AB)C = A(BC)$$
$$(A \cup B) \cup C = A \cup (B \cup C) \qquad (A+B)+C = A+(B+C)$$

We can then summarize these laws as follows.

Law	Logic	Sets	Boolean Algebra
Identity	$p \wedge 1 \Leftrightarrow p$	$A \cap U = A$	$A \cdot 1 = A$
	$p \vee 1 \Leftrightarrow 1$	$A \cup U = U$	$A+1 = 1$
	$p \wedge 0 \Leftrightarrow 0$	$A \cap \varnothing = \varnothing$	$A \cdot 0 = 0$
	$p \vee 0 \Leftrightarrow p$	$A \cup \varnothing = A$	$A+0 = A$
Idempotent	$p \wedge p \Leftrightarrow p$	$A \cap A = A$	$AA = A$
	$p \vee p \Leftrightarrow p$	$A \cup A = A$	$A+A = A$
Complement	$\overline{\overline{p}} \Leftrightarrow p$	$\overline{\overline{A}} = A$	$\overline{\overline{A}} = A$
	$p \wedge \overline{p} \Leftrightarrow 0$	$A \cap \overline{A} = \varnothing$	$A\overline{A} = 0$
	$p \vee \overline{p} \Leftrightarrow 1$	$A \cup \overline{A} = U$	$A + \overline{A} = 1$
Commutative	$p \wedge q \Leftrightarrow q \wedge p$	$A \cap B = B \cap A$	$AB = BA$
	$p \vee q \Leftrightarrow q \vee p$	$A \cup B = B \cup A$	$A + B = B + A$
Associative	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	$(A \cap B) \cap C = A \cap (B \cap C)$	$(AB)C = A(BC)$
	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A+B)+C = A+(B+C)$

How, then, can we use these laws?

Simplifying Logical Expressions

We can now use these laws to simplify logical expressions or to prove logical equivalence without resorting to truth tables.

Suppose we wish to simplify $(p \wedge 1) \vee (\overline{q} \wedge 0) \vee (\overline{r} \wedge r)$. Note that this would require a truth table with 8 rows to show all combinations of p, q, and r. However, to do so using the laws of logic will require fewer steps.

The procedure for simplifying an expression using the laws of logic is to simplify each piece of the expressing using a single law, then write the name of the law you are using to one side (writing the name of the law is **required**, and **not optional**!). If you are using more than one law, then use a separate line for each law/step.

Simplifying $(p \wedge 1) \vee (\bar{q} \wedge 0) \vee (\bar{r} \wedge r)$ would then give

$(p \wedge 1) \vee (\bar{q} \wedge 0) \vee (\bar{r} \wedge r)$

$p \vee 0 \vee (\bar{r} \wedge r)$	identity
$(p \vee 0) \vee (\bar{r} \wedge r)$	associative
$p \vee (\bar{r} \wedge r)$	identity
$p \vee (r \wedge \bar{r})$	commutative
$p \vee 0$	complement
p	identity

Our conclusion is therefore that $(p \wedge 1) \vee (\bar{q} \wedge 0) \vee (\bar{r} \wedge r) \Leftrightarrow p$.

We could also do an alternate solution, using a different order of steps to get our answer.

$(p \wedge 1) \vee (\bar{q} \wedge 0) \vee (\bar{r} \wedge r)$

$p \vee 0 \vee (\bar{r} \wedge r)$	identity
$p \vee 0 \vee 0$	complement
$p \vee 0$	definition of "or"
p	identity

And we reach the same conclusion.

Example

Simplify $(p \vee \bar{p}) \wedge (\bar{p} \vee \bar{p})$.

Answer:

$(p \vee \bar{p}) \wedge (\bar{p} \vee \bar{p})$

$1 \wedge (\bar{p} \vee \bar{p})$	complement

1 ^($\bar{p}$) idempotent

$\bar{p}$ identity

(And if you applied the laws correctly but in a different order or combination, you should still come to the same, correct conclusion.)