Section 3.1: Sequences and Series

Sequences

Let's start out with the definition of a sequence:

sequence: an ordered list of numbers, often with a definite pattern

Recall that in a set, order doesn't matter so this is one way that a sequence differs from a set. Also, repetition doesn't matter in a set but does in a sequence: if a number is repeated in a sequence, it isn't considered a "duplicate" and cannot be removed without changing the sequence.

Sequences, like sets, can be finite or infinite. If a sequence is finite, then either the last term or the number of terms must be specified so that it's clear where the sequence stops.

Example

Which of the following sequences are infinite? Which are finite?

a) 7, 11, 15, 19, …

b) 1, 4, 9, 16, 25, 36, … 100

c) $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ... \frac{1}{256}$

Answer

b) and c) are finite, because their last terms are given. a), however, goes on forever so is infinite.

To begin with, let's examine some sequences in detail. We will begin by looking for patterns in each sequence.

Example

What is the pattern for the following sequences? What is the next term for each sequence?

a) 7, 11, 15, 19, …

b) 1, 4, 9, 16, 25, 36, … 100

c) $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ... \frac{1}{256}$

d) 3, –6, 12, –24, …

e) 3, –6, –15, –24, …

Answer

a) The pattern is that you add 4 to the previous term to get the next term. The next term is then 23.

b) The pattern is that if you say that "1" is the first term and "4" is the second term, then n^2 will be the nth term. So the next term after 36 is 49.

c) The pattern is to divide each term by two (or multiply by a half) to get the next term. So the term after 1/16 will be 1/32.

d) The pattern is to multiply each term by –2 to get the next term. The next term is then 48.

e) The pattern is to subtract 9 from the previous term, so the next one is –33.

Note that in this previous example, the last two sequences looked very similar for three of their first four terms. However, the third term is different so the pattern for the two sequences is not the same and subsequent terms could look very different.

Notation

We will use the notation a_n for the nth term in a sequence, where n is the index. For example, the first term would then be a_1, the second term a_2, and so on. The index n, then, is a positive integer (or a natural number, if you like).

Other notations may start their counting with a_o being the first term. For the purposes of this course, we'll stick to starting at $n = 1$.

Defining a Sequence

There are three ways to define a sequence:

1) List all of the terms, or enough terms to set up the pattern. If the sequence is finite, then either the last term or the number of terms must be given.

2) Give a general formula for the nth term.

3) Give a recursive formula for the nth term.

Let's look at examples of each type. For instance, the sequences 7, 11, 15, 19, … and 1, 4, 9, 16, 25, 36, … 100 are examples of sequences defined by listing the terms.

General Formula

A general formula is a formula that gives a_n as a function of n only. Let's look at the following examples to examine some sequences defined in this way.

Example

Give the first four terms of the sequence given by the general formula $a_n = 4n + 3$.

Answer
$a_n = 4n + 3$, so

$a_1 = 4\times1 + 3 = 7$

$a_2 = 4\times2 + 3 = 11$

$a_3 = 4\times3 + 3 = 15$

$a_4 = 4\times4 + 3 = 19$

The first four terms are then 7, 11, 15, and 19. This is the same sequence that was given as part a) in the first two examples of this section.

Example

Give all terms of the sequence given by the formula $a_n = \left(\frac{1}{3}\right)^n$ for $1 \le n \le 5$.

Answer

This is a finite sequence, since restrictions have been placed on the values of n. The terms are then:

$$a_1 = \left(\frac{1}{3}\right)^1 = \frac{1}{3}$$

$$a_2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$a_3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

$$a_4 = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$$

$$a_5 = \left(\frac{1}{3}\right)^5 = \frac{1}{243}$$

You can see from the previous examples that the general formula allows you to calculate a_n for any value of n. The very useful thing about the general formula is that you don't need to know the previous term to calculate a particular term. For instance, if you want to know the 50th term of the sequence 7, 11, 15, 19, …, you can determine that the pattern is to add 4 to the previous term to get the next term. However, to get the 50th term, you'd have to calculate the 49th first, but the 49th requires the 48th, and so on. But if you instead use the expression $a_n = 4n+3$, which gives the same sequence, then the 50th term is just

$$a_n = 4n+3$$
$$a_{50} = 4\cdot 50+3 = 203$$

and there's no need to calculate preceding terms. Handy!

Recursive Definition

A recursive formula gives a formula for the next term in terms of the previous one. For example, in our old friend 7, 11, 15, 19, … , the next term is found by adding 4 to the previous term: $a_n = a_{n-1}+4$. However, that's not enough information to uniquely define the series because you don't know where to start. A complete definition must include the first term also. Therefore, the recursive definition for our old friend 7, 11, 15, 19, … would be

$$\begin{cases} a_1 = 7 \\ a_n = a_{n-1}+4 \end{cases}$$

Recursive definitions, then, must specify the first term or terms and also the rule which allows you to calculate the next term from the previous term or terms.

Example

Calculate the first four terms of the sequence given by

$$\begin{cases} a_1 = 3 \\ a_n = \left(a_{n-1}-1\right)^2+10 \end{cases}$$

Answer

The first term is already given, $a_1 = 3$. Then

$$a_2 = \left(3-1\right)^2+10 = 2^2+10 = 14$$
$$a_3 = \left(14-1\right)^2+10 = 13^2+10 = 179$$
$$a_4 = \left(179-1\right)^2+10 = 178^2+10 = 31694$$

Example

Give a recursive formula for the sequence 2, 6, 18, 54, …

Answer

The pattern is that the next term equals the previous term times three. Therefore,

$$\begin{cases} a_1 = 2 \\ a_n = 3a_{n-1} \end{cases}$$

Recursive definitions have the same drawback that we've seen before: if we want to know the 200th term, we need to calculate the 199th first, and so on. Only the general formula allows us to calculate each term directly without knowing the previous one.

Fibonacci sequence

The Fibonacci sequence is the most famous example of a recursive sequence:

1, 1, 2, 3, 5, 8, 13, …

The pattern can be quite difficult to spot – you get the next term from the **sum** of the two previous terms. The recursive formula for this sequence is therefore

$$\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \end{cases}$$

Here, the first two terms must be given to start off with so that you are then able to calculate the third term from the previous two.

Series

A series is the sum of a sequence, which can be finite or infinite. Here are two examples:

a) 16 + 20 + 24 + 28 + … 64

b) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$

Notation

The sum of the first n terms of a sequence is denoted by S_n (also sometimes called the nth partial sum). If the series is finite, it could be the sum of **all** of the terms. S_∞ is how we write the sum of an infinite series, like the second example above.

Example

For the series 16 + 20 + 24 + 28 + … 64, calculate S_3 and S_4.

Answer

$S_3 = 16 + 20 + 24 = 60$

$S_4 = 16 + 20 + 24 + 28 = 88$

However, it's easy to see that this method becomes very cumbersome for large values of n. We'll develop some more efficient methods in the next two sections.

Sigma notation

It's easy to take a sequence in list form and transform it into a series by changing all of the commas to + signs. However, what if you are given the general formula instead? For example, let's take 7, 11, 15, 19, … which we know to be $a_n = 4n + 3$. Since the general form is so useful for finding a_n when n is large, it would be nice if we could retain that information while writing our sum.

To do so, we'll introduce a new notation called "sigma notation". It uses the Greek letter sigma (the uppercase one): Σ, which is commonly used to mean "sum of".

Let's look at an example of sigma notation and discuss what all of the parts mean. Consider the following

$$\sum_{i=1}^{5}(4i+3)$$

The letter i is an index here, and it runs from the value given at the bottom of the sigma to the number at the top of the sigma in steps of 1. Here, i runs from 1 to 5. We are summing, then, the value of $4i + 3$ for each value of i as it runs from 1 to 5:

$$\begin{aligned}
& \qquad\quad i=1 \qquad\quad i=2 \qquad\quad i=3 \qquad\quad i=4 \qquad\quad i=5 \\
\sum_{i=1}^{5}(4i+3) &= \left(4\times1+3\right)+\left(4\times2+3\right)+\left(4\times3+3\right)+\left(4\times4+3\right)+\left(4\times5+3\right) \\
&= \quad 7 \quad + \quad 11 \quad + \quad 15 \quad + \quad 19 \quad + \quad 23 \\
&= 75
\end{aligned}$$

Let's look at more examples.

Example

Calculate $\sum_{i=1}^{3}(2i-5)$

Answer

$$\begin{array}{l} \quad\quad\quad\quad\quad\quad\quad\quad i=1 \quad\quad\quad i=2 \quad\quad\quad i=3 \\ \sum_{i=1}^{3}(2i-5)=\left(2\times1-5\right)+\left(2\times2-5\right)+\left(2\times3-5\right) \\ \quad\quad\quad\quad\quad = \quad -3 \quad + \quad -1 \quad + \quad 1 \\ \quad\quad\quad\quad\quad =-3 \end{array}$$

Example

Calculate $\sum_{j=6}^{9}\left(8-j\right)^2$

Answer

$$\begin{array}{l} \quad\quad\quad\quad\quad\quad\quad j=6 \quad\quad j=7 \quad\quad j=8 \quad\quad j=9 \\ \sum_{j=6}^{9}\left(8-j\right)^2=\left(8-6\right)^2+\left(8-7\right)^2+\left(8-8\right)^2+\left(8-9\right)^2 \\ \quad\quad\quad\quad\quad = \quad 4 \quad + \quad 1 \quad + \quad 0 \quad + \quad 1 \\ \quad\quad\quad\quad\quad =6 \end{array}$$

Example

Calculate $\sum_{k=12}^{16}3$

Answer

$$\begin{array}{l} \quad\quad k=12 \quad\quad k=13 \quad\quad k=14 \quad\quad k=15 \quad k=16 \\ \sum_{j=12}^{16}3= \quad 3 \quad + \quad 3 \quad + \quad 3 \quad + \quad 3 \quad + \quad 3 \\ \quad\quad = \; 15 \end{array}$$

The tricky thing about the last one is deciding how many terms there are. You may, as is shown above, write out all of the possible values of the index. Or you may use the following nifty rule:

$$\# \text{ terms} = \text{last} - \text{first} + 1$$

For instance, the last example had the index running from 12 to 16. The number of terms, then, for that series is 16 – 12 + 1 = 5.

Example

Write the following series in sigma notation:

$$4+9+16+25+\ldots 100$$

Answer

Let's pick our index first. If we want to be lazy, instead of starting our index at 1, we could start at 2 and our series would be

$$\sum_{k=2}^{10} k^2$$

Other acceptable answers would involve changing our starting point for the index to give $\sum_{j=1}^{9}(j+1)^2$ or $\sum_{i=0}^{8}(i+2)^2$ or even $\sum_{l=157}^{165}(l-155)^2$ if 157 happens to be your favourite number.

Example

Write the following sequence in sigma notation:

$$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots$$

Answer

$$\sum_{j=3}^{\infty}\frac{1}{j}$$

To write an infinite series in sigma notation, you just replace the final value of the index with ∞.