Section 3.2: Arithmetic Sequences and Series

Arithmetic Sequences

Let's start out with a definition:

> arithmetic sequence: a sequence in which the next term is found by adding a constant (the common difference *d*) to the previous term

Here are some examples of arithmetic sequences:

a) 7, 11, 15, 19, …

b) 11, 4, –3, –10, … –59

c) 12, 12.3, 12.6, 12.9, …

The first one has a common difference of 4, the second –7, and the third 0.3. Note that in each of them, we can find the common difference *d* by taking **any** term and subtracting the previous term from it.

Example

For the following sequences, state whether each of them is arithmetic.

a) –3, –10, –17, –24, …

b) 4, 5, 7, 10, …

c) 2, 4, 8, 16, …

d) $\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots\frac{1}{20}$

Answer

a) Yes, because the common difference *d* is –7.

b) No, because you're not adding the same number each time.

c) No, because you're multiplying by 2 to get the next term, not adding.

d) No, because the difference between each pair of terms is different.

Again, you can define an arithmetic sequence in one of three ways: by listing the terms, by giving a recursive definition, or by giving a general definition.

Recursive Definitions for Arithmetic Sequences

Let's look first at an example.

Example

Give a recursive definition for the sequence 2, 10, 18, 26, …

Answer

Recall that a recursive definition has two parts: listing the first term and giving the pattern. In this case, the pattern is adding d = 8 to the previous term to get the next term. The recursive definition is therefore

$$\begin{cases} a_1 = 2 \\ a_n = a_{n-1} + 8 \end{cases}$$

More generally, the recursive formula for **any** arithmetic sequence is

$$\begin{cases} a_1 = \text{< insert value here>} \\ a_n = a_{n-1} + d \end{cases}$$

General Formulae for Arithmetic Sequences

Let's examine the previous example in more detail to see if we can recognize any patterns and come up with a general formula. Rewriting each term, we get

$$\begin{array}{l} 2, \quad 10, \quad 18, \qquad 26, \quad ... \\ 2, 2+6, 2+6\times 2, 2+6\times 3, ... \end{array}$$

So the 3rd term equals the first plus 6 times 2, the 4th term equals the first plus 6 times 3, and the nth term will equal the first plus 6 times (n – 1). More generally, the nth term will equal the first plus d times (n – 1). In other words,

$$a_n = a_1 + (n-1)d$$

for any **arithmetic sequence**.

Example

Write a general formula for the sequence 2, 10, 18, 26, …

Answer

This sequence is arithmetic with the first term 2 and common difference 8.

$$
\begin{aligned}
a_n &= a_1 + (n-1)d \\
a_n &= 2 + (n-1)8 \\
a_n &= 2 + 8n - 8 \\
a_n &= 8n - 6
\end{aligned}
$$

The general formula is then that $a_n = 8n - 6$.

Example

What is the 50th term in the sequence in the sequence 2, 10, 18, 26, … ?

Answer

This is the same sequence from the previous example. We may then use the formula we derived, $a_n = 8n - 6$, with $n = 50$.

$$
\begin{aligned}
a_n &= 8n - 6 \\
a_n &= 8 \times 50 - 6 \\
a_n &= 400 - 6 \\
a_n &= 394
\end{aligned}
$$

The 50th term is 394.

Example

What is the common difference in the arithmetic sequence in which the first term is 18 and the twelfth term is –59?

Answer

$$
\begin{aligned}
a_n &= a_1 + (n-1)d \\
-59 &= 18 + (12-1)d \\
-77 &= 11d \\
d &= -7
\end{aligned}
$$

The common difference is –7.

Example

Which term has a value of 404 in the sequence –37, –28, –19, …?

Answer

So a_1 is –37 and d is +9. Then we want to find the value of n for which a_n equals 404.

$$a_n = a_1 + (n-1)d$$
$$404 = -37 + (n-1)9$$
$$441 = 9(n-1)$$
$$49 = n-1$$
$$n = 50$$

The **fiftieth** term is 404.

Example

Find the first four terms of the arithmetic sequence in which the thirteenth term is 97 and the fiftieth term is 393.

Answer

So a_{13} = 97 and a_{50} = 393.

Then we find that

$$a_n = a_1 + (n-1)d$$
$$97 = a_1 + (13-1)d$$
$$97 = a_1 + 12d$$

However, this has two unknowns, a_1 and d. Let's look at a_{50}:

$$a_n = a_1 + (n-1)d$$
$$393 = a_1 + (50-1)d$$
$$393 = a_1 + 49d$$

We now have two unknowns, but two equations, giving us the system

$$\begin{cases} 97 = a_1 + 12d \\ 393 = a_1 + 49d \end{cases}$$

Solving this system, we first multiply the top equation by negative 1:

$$-97 = -a_1 - 12d$$
$$393 = a_1 + 49d$$

And then add the two equations together, so that the a_1 terms cancel out.

$$296 = 37d$$
$$d = 8$$

Now we substitute into one of the original equations:

$$97 = a_1 + 12d$$
$$97 = a_1 + 12 \times 8$$
$$97 = a_1 + 96$$
$$a_1 = 1$$

Since $a_1 = 1$ and $d = 8$, our sequence is then 1, 9, 17, 25, …

Arithmetic Series

Recall that S_n is the sum of the first n terms of a series. Let's look at a couple of examples of arithmetic series to see if we can identify any patterns.

Suppose we wish to take some partial sums of the series 2 + 10 + 18 + 26 + …. Let's first calculate S_6. We could just find the first six terms and add them up, but notice the following:

$$S_6 = 2 + 10 + 18 + 26 + 34 + 42$$

The sum of the first and last numbers is 44. The sum of the second and second-to-last is also 44. So is the sum of the third and third-last. So when you take the terms in pairs, each pair has the same sum, $(a_1 + a_n)$, and there are $n/2$ pairs in total. Then

$$S_n = \frac{n}{2}(a_1 + a_n).$$

What if, however, there are an odd number of terms? Let's also calculate S_7:

$$S_7 = 2 + 10 + 18 + 26 + 34 + 42 + 50$$

The sum of the first and last is 52, as is the sum of the each "inner pair". Notice that the middle, unpaired value, is ½ of 52. So in a sense, the middle term is ½ of a pair, for a total of 3½ pairs. But that's just 7/2, which is our $n/2$ in the original formula! So we're still good. The relationship

$$S_n = \frac{n}{2}(a_1 + a_n)$$

still works, for both odd and even values of n.

Example

Find the sum of the first forty terms of the series 2 + 10 + 18 + 26 + …. .

Answer

This is just the same sequence as before, with $a_1 = 2$ and $d = 8$. In order to use our previous formula, however, we need to calculate a_{40} before we can calculate S_{40}.

$$a_n = a_1 + (n-1)d$$
$$a_{40} = 2 + 39 \times 8 = 314$$

So,

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{40} = \frac{40}{2}(2 + 314) = 20 \times 316 = 6320$$

The sum of the first forty terms is 6320. (Much easier than writing out the first forty terms and adding them up!)

In the previous example, we used the formula for a_n to calculate the last term and put its value into the formula for S_n. We could do that in a more general way:

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) \\ &= \frac{n}{2}\left[a_1 + \left(a_1 + (n-1)d\right)\right] \\ &= \frac{n}{2}\left[2a_1 + (n-1)d\right] \end{aligned}$$

and the last expression, which gives S_n as a function of the first term, the number of terms, and the common difference, can also be used to evaluate series.

Example

Find the sum of the first one hundred terms of the sequence 5, –6, –17, –26, … .

Answer

This sum will just be 5 + –6 + –17 + –26 + …, with a_1 = 5 and d = –11.

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$S_{100} = \frac{100}{2}\left[2\times5+99\times(-11)\right] = -53950$$

Example

Calculate $\sum_{j=3}^{18} 5n+10$.

Answer

The first term will be for j=3 and will equal 5(3)+10=25. Next is j=4 and will equal 5(4)+10=30, j=5 equaling 5(5)=35, and so on.   The last term will be for j=18 and will equal 5(18)+10=100.

In other words, our series is 25+30+35+…100. Is it arithmetic? Yes, with common difference d = 5.

What else do we need for our calculation? The number of terms equals (last – first +1), so is (18-3+1)=16. Then

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{16} = \frac{16}{2}(25+100) = 1000$$

Example

Pat the math instructor asks her students to do five word problems the first week, six the second week, seven the third week, and so on, increasing the number of word problems each week by one.

a) How many word problems will diligent students be doing in the last week of classes (the 11^{th} week)?

b) How many word problems will diligent students have completed during the course of the term (11 weeks)?

Answer

a) The number of word problems is a sequence: 5, 6, 7, … . In fact, it's an arithmetic sequence with $a_1 = 5$ and $d = 1$. In the eleventh week, then,

$$a_n = a_1 + (n-1)d$$
$$a_{11} = 5 + 10 \times 1 = 15$$

Diligent students will solve 15 word problems in the last week of classes.

b) The **total** number of word problems solved is

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{11} = \frac{11}{2}(5 + 15) = 110$$

Diligent students will have solved 110 word problems in total.

Summary

For an **arithmetic** sequence, the *n*th term is given by $a_n = a_1 + (n-1)d$

For an **arithmetic** series, the sum of the first n terms (nth partial sum) is $S_n = \frac{n}{2}(a_1 + a_n)$ or $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$