# Section 3.3: Geometric Sequences and Series

## Geometric Sequences

Let's start out with a definition:

geometric sequence: a sequence in which the next term is found by multiplying the previous term by a constant (the common ratio *r*)

Here are some examples of geometric sequences:

a) 9, 18, 36, 72, …

b) 12, 18, 27, $\frac{81}{2}$,…

c) 10, –30, 90, –270, … –196830

d) –3, –12, –48, –192, …

e) 48, –36, 27, …

The common ratios of each of these sequences, in order from a) to e), is 2, $\frac{3}{2}$, –3, 4, $-\frac{3}{4}$, respectively. Note that in each of them, we can find the common ratio *r* by taking **any** term and dividing it by the previous term.

Like any other sequences, geometric sequences can be finite or infinite. Example c) above is finite, as the last term is specified. The others are infinite sequences.

*Example*

For each of the following sequences, state whether it is arithmetic, geometric, or neither.

a) 45, 15, 5, …

b) 5, 3, 1, –1, …

c) 1, 8, 27, 64, … , 1000

d) –1, 1, –1, 1, –1, 1, …

Answer

a) Geometric, because the common ratio *r* is $\frac{1}{3}$.

b) Arithmetic, because the common difference $d$ is –2.

c) Neither, because there isn't either a common difference or ratio between terms. (In fact, the pattern is that $a_n = n^3$.)

d) Geometric, because the common ratio $r$ is –1.

Again, you can define a geometric sequence in one of three ways: by listing the terms, by giving a recursive definition, or by giving a general definition.

**Recursive Definitions for Geometric Sequences**

Let's look at an example.

*Example*

Give a recursive definition for the sequence 2, 10, 50, 250, …

Answer

Recall that a recursive definition has two parts: listing the first term and giving the pattern. In this case, the pattern is multiplying the previous term by $r = 5$ to get the next term. The recursive definition is therefore

$$\begin{cases} a_1 = 2 \\ a_n = 5a_{n-1} \end{cases}$$

More generally, the recursive definition for **any** geometric sequence is

$$\begin{cases} a_1 = \text{<insert value here>} \\ a_n = a_{n-1} \times r \end{cases}$$

**General Formulae for Geometric Sequences**

Let's examine the previous example in more detail to see if we can recognize any patterns and come up with a general formula. Rewriting each term, we get

$$2,\ 10,\ 50,\ 250,\ \ldots$$
$$2,\ 2\times 5,\ 2\times 5^2,\ 2\times 5^3,\ \ldots$$

So the 3rd term equals the first times 5 squared, the 4th term equals the first times 5 cubed, and the $n$th term will equal the first times 5 raised to the ($n$ – 1) power. More generally, the $n$th term equals the first term times $r$ raised to the ($n$ – 1) power, namely

$$a_n = a_1 r^{n-1}$$

for all **geometric** sequences.

*Example*

Write a general formula for the sequence 3, 6, 12, …

Answer

This sequence is geometric with the first term 3 and common ratio 2.

$$a_n = a_1 r^{n-1}$$
$$a_n = 3 \times (2)^{n-1}$$

The general formula is then that $a_n = 3 \times 2^{n-1}$.

*Example*

What is the 20th term in the sequence in the sequence 3, 6, 12, … ?

Answer

This is the same sequence from the previous example. We may then use the formula we derived above with $n = 20$.

$$a_n = a_1 r^{n-1}$$
$$a_{20} = 3 \times 2^{20-1}$$
$$a_{20} = 3 \times 2^{19}$$
$$a_{20} = 1,572,864$$

The 20th term is 1,572,864, which provides a nice example for how fast geometric sequences can grow, even for small values of $r$.

*Example*

Write a general formula for the sequence 8, 12, 18, 27, … ? What is the fifteenth term in this sequence? The fiftieth?

Answer

$$a_n = a_1 r^{n-1}$$
$$a_n = 8\left(\frac{3}{2}\right)^{n-1}$$
$$a_{15} = 8\left(\frac{3}{2}\right)^{14} \approx 2335.43$$
$$a_{50} = 8\left(\frac{3}{2}\right)^{49} \approx 3.40065 \times 10^9$$

So the general formula is $a_n = 8\left(\frac{3}{2}\right)^{n-1}$ and the fifteenth and fiftieth terms are 2335.43 and $3.4 \times 10^9$, respectively.

**Geometric Series**

Recall that $S_n$ is the sum of the first $n$ terms of a series. Let's look at how a formula for $S_n$ is derived.

$$S_n = a_1 + a_2 + a_3 + a_4 + \dots + a_{n-2} + a_{n-1} + a_n$$
$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \dots + a_1 r^{n-3} + a_1 r^{n-2} + a_1 r^{n-1}$$

Let's take that last expression for $S_n$ and multiply it by $-r$ to get

$$-rS_n = -a_1 r - a_1 r^2 - a_1 r^3 - a_1 r^4 - \dots - a_1 r^{n-2} - a_1 r^{n-1} - a_1 r^n$$

Then if we add the rows for $S_n$ and $-rS_n$ , we get

$$\begin{array}{l} S_n = \quad a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \dots + a_1 r^{n-3} + a_1 r^{n-2} + a_1 r^{n-1} \\ -rS_n = -a_1 r - a_1 r^2 - a_1 r^3 - a_1 r^4 - \dots - a_1 r^{n-2} - a_1 r^{n-1} - a_1 r^n \\ \hline S_n - rS_n = a_1 - a_1 r^n \end{array}$$

since all of the terms in between these two ($a_1$ and $a_1 r^n$) will cancel. Then

$$S_n(1-r) = a_1(1-r^n)$$

and

$$S_n = \frac{a_1(1-r^n)}{(1-r)}$$

The last formula above is the formula for the sum of the first $n$ terms for **any geometric series**.

*Example*

Find the sum of the first 20 terms of the series 3 + 6 + 12 + …

Answer

This is a geometric series with $a_1 = 3$ and $r = 2$. We want to find $S_{20}$.

$$S_n = \frac{a_1\left(1-r^n\right)}{\left(1-r\right)}$$

$$S_{20} = \frac{3\left(1-2^{20}\right)}{\left(1-2\right)} = 3{,}145{,}725$$

The sum of the first 20 terms is 3,145,725.

*Example*

Find the sum of the first forty terms of the series 8 – 12 + 18 – 27 ….

Answer

This is a geometric series with $a_1 = 8$ and $r = -\frac{3}{2}$. We want to find $S_{40}$.

$$S_n = \frac{a_1\left(1-r^n\right)}{\left(1-r\right)}$$

$$S_{20} = \frac{8\left(1-\left(-1.5\right)^{40}\right)}{\left(1-\left(-1.5\right)\right)} = -3.53835\times 10^7$$

The sum of the first forty terms is $-3.54 \times 10^7$.

## Sum of an Infinite Geometric Series

Let's take a look at the infinite series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$

What happens when we try to evaluate this sum using the $S_n$ formula? We can put $a_1 = ½$, $r = ½$, and $n = \infty$ into the formula, but we will run into a roadblock when we try to evaluate $(½)^{\infty}$.

Let's take a closer look at the behaviour of $(½)^n$ for large values of $n$. As $n$ gets larger, the fraction $\left(\frac{1}{2}\right)^n = \frac{1}{2^n}$ gets ever smaller. In fact, as $n$ approaches ∞, $(½)^n$ will approach zero.

This is true for any $r$ provided that $|r| < 1$. (If you're not familiar with the absolute value bars, $|x|$, an equivalent expression is that $-1 < r < 1$.)

Recalling that $S_n = \frac{a_1\left(1-r^n\right)}{\left(1-r\right)}$ and letting the $r^n$ term go to zero, then

$$S_\infty = \frac{a_1}{1-r} \text{ for } -1 < r < 1$$

for any **infinite geometric series**, provided that $r$ meets the restriction above.

Let's now revisit the series that started this discussion, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$, and evaluate it in the following example.

*Example*

Evaluate $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$.

Answer

This series is geometric with $a_1 = ½$ and $r = ½$. Then

$$S_\infty = \frac{a_1}{1-r} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$$

The sum of this series is 1.

*Example*

Evaluate $24 + 16 + \frac{32}{3} + \ldots$

Answer

This series is geometric with $a_1 = 24$ and $r = \frac{2}{3}$.

$$S_\infty = \frac{a_1}{1-r} = \frac{24}{1-\frac{2}{3}} = \frac{24}{\frac{1}{3}} = 24 \times \frac{3}{1} = 72$$

*Example*

Evaluate 24 – 16 + $\frac{32}{3}$ + …

Answer

This series is identical to the previous one except that $r$ is now negative: $a_1 = 24$ and $r = -\frac{2}{3}$.

$$S_\infty = \frac{a_1}{1-r} = \frac{24}{1-\left(-\frac{2}{3}\right)} = \frac{24}{1+\frac{2}{3}} = \frac{24}{\frac{5}{3}} = 24 \times \frac{3}{5} = \frac{72}{5} = 14.4$$

*Example*

Evaluate 12 + 18 + 27 + …

Answer

This series is geometric with $a_1 = 12$ and $r = \frac{3}{2}$. You may already realize what's going on, but in case you don't, let's naively put the values into the formula and see what we get:

$$S_\infty = \frac{a_1}{1-r} = \frac{12}{1-\frac{3}{2}} = \frac{12}{-\frac{1}{2}} = 12 \times -\frac{2}{1} = -24$$

Wait! How can the sum of a bunch of positive number be negative? The answer is that our restriction for $r$ is that it must be between –1 and 1, but $r$ = 1.5. Because $r$ does not satisfy the restriction, we cannot use the above formula for $S_\infty$. Indeed, if you add up a bunch of positive numbers that are increasing as you go up, you can see that the sum just keeps getting bigger as we add more terms. You could then either say that the sum is infinite (dicey) or "does not exist" (safer).

But why is it safer to say "does not exist" in the last example? Let's look at three sums:

a) 12 + 18 + 27 + …

b) $-12 - 18 - 27 - \ldots$

c) $12 - 18 + 27 + \ldots$

Each term in a) is getting more positive, so the sum of that sequence will be $+\infty$. Each term in b) is getting more and more negative, so the sum of that sequence will be $-\infty$. But in the last term, the sum oscillates back and forth: $S_1 = 12$, $S_2 = -6$, $S_3 = 21$, $S_4 = -19.5$, and so on. The sign of $S_n$ is either positive or negative depending on whether the number of terms you've added is even or odd. Rather than debating whether infinity is odd or even (?!), we will just say that the sum "does not exist."

*Example*

Evaluate $\sum_{j=0}^{\infty} 27\left(\frac{1}{3}\right)^j$.

Answer

Ick! The best place to start is to figure out the first few terms to determine the pattern:

when $j = 0$, $27\left(\frac{1}{3}\right)^0 = 27 \times 1 = 27$

when $j = 1$, $27\left(\frac{1}{3}\right)^1 = 27 \times \frac{1}{3} = 9$

when $j = 2$, $27\left(\frac{1}{3}\right)^2 = 27 \times \frac{1}{3^2} = 3$

so our sequence is 27, 9, 3, … This is geometric with $a_1 = 27$ and $r = \frac{1}{3}$. Then

$$S_\infty = \frac{a_1}{1-r} = \frac{27}{1-\frac{1}{3}} = \frac{27}{\frac{2}{3}} = 27 \times \frac{3}{2} = \frac{81}{2} = 40.5$$

*Example*

Evaluate $\sum_{k=5}^{\infty} \frac{1}{2}k$.

Answer

Once again, let's figure out the first few terms to determine the pattern:

when $k = 5$, $\frac{1}{2}k = \frac{1}{2}5 = 2.5$

when $k = 6$, $\frac{1}{2}k = \frac{1}{2}6 = 3$

when $k = 7$, $\frac{1}{2}k = \frac{1}{2}7 = 3.5$

so our sequence is 2.5, 3, 3.5. Wait! This is arithmetic! Not only that, but the numbers are increasing. So the sum will be infinite, or if you prefer, the sum "does not exist".

**Repeating Decimals**

Let's examine $0.\overline{7}$ in some detail to see what we find:

$$\begin{aligned} 0.\overline{7} &= 0.777777777... \\ &= 0.7 + 0.07 + 0.007 + 0.0007 + ... \end{aligned}$$

But this is just the sum of an infinite series with $a_1 = 0.7$ and $r = 0.1$. Rewriting $a_1$ and $r$ in fraction form (you'll see why in a minute) gives $a_1 = \frac{7}{10}$ and $r = \frac{1}{10}$. Then

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{10} \times \frac{10}{9} = \frac{7}{9}$$

So $0.\overline{7} = 7/9$. Interesting!

*Example*

Find an exact fraction for $0.\overline{6}$.

Answer

$$\begin{aligned} 0.\overline{6} &= 0.66666666... \\ &= 0.6 + 0.06 + 0.006 + 0.0006 + ... \end{aligned}$$

But this is just the sum of an infinite series with $a_1 = \frac{6}{10}$ and $r = \frac{1}{10}$. Then

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{6}{10}}{1-\frac{1}{10}} = \frac{\frac{6}{10}}{\frac{9}{10}} = \frac{6}{10} \times \frac{10}{9} = \frac{6}{9} = \frac{2}{3}$$

So $0.\overline{6} = 2/3$.

*Example*

Find an exact fraction for $0.\overline{18}$.

Answer

$$\begin{aligned} 0.\overline{18} &= 0.1818181818... \\ &= 0.18 + 0.0018 + 0.000018 + ... \end{aligned}$$

But this is just the sum of an infinite series with $a_1 = \frac{18}{100}$ and $r = \frac{1}{100}$. Then

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{18}{100}}{1-\frac{1}{100}} = \frac{\frac{18}{100}}{\frac{90}{100}} = \frac{18}{100} \times \frac{100}{99} = \frac{18}{99} = \frac{2}{11}$$

So $0.\overline{18} = 2/11$.

**Summary**

For a **geometric** sequence, the *n*th term is given by $a_n = a_1 r^{n-1}$

For a **geometric** series, the sum of the first *n* terms (*n*th partial sum) is $S_n = \frac{a_1\left(1-r^n\right)}{\left(1-r\right)}$

For an **infinite geometric** series, the sum is $S_\infty = \frac{a_1}{1-r}$, provided that $-1 < r < 1$.