Section 6.1: Counting Techniques

Although the idea of counting the number of objects seems straightforward, there are a few tricky bits to it when you are looking at large quantities. Let's start by looking at an example.

Example

How many three-digit natural numbers are there?

Answer: Let's look at the list: 100, 101, 102, … 999. There are three methods we can use to determine the total number here.

Method #1: We could, for example, write the list as a sequence. We see that it's an arithmetic sequence with $d = 1$. Then

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ 999 &= 100 + (n-1)1 \\ 899 &= n-1 \\ n &= 900 \end{aligned}$$

so there are 900 three-digit numbers in total.

Method #2: Since the numbers go up one-by-one, can use the nice summation notation trick of (last – first + 1) = (999 – 100 + 1) = 900 as well.

Method #3: Consider the digits ___ ___ ___ . The first digit can be 1-9 for 9 choices, the second and third can be 0-9 for 10 choices. Then you get $9 \times 10 \times 10 = 900$ numbers.

Example

How many three-digit natural numbers are divisible by 5?

Answer: The numbers are 100, 105, 110, … 995

Method #1 works very nicely:

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ 995 &= 100 + (n-1)5 \\ 895 &= (n-1)5 \\ 179 &= n-1 \\ n &= 180 \end{aligned}$$

Method #2: we can rewrite the sequence as 20×5, 21×5, 22×5, …199×5. Then we use (last – first + 1) = (199 – 20 + 1) = 180. (We can only use this formula when we are counting by steps of one. So we have to force our sequence into a counting-by-one step to use it.)

Method #3: We note that if the number is divisible by 5, then the last digit is either 0 or 5, for two choices. We then get 9×10×2 = 180. However, be careful with this method! It will work well for numbers divisible by 1, 2, 5, and 10, because this eliminates digits in the last column. You can't use this method at all with most other divisors like 3, 4, 6, etc.

The reason Method #3 works is called the Multiplication Principle of Counting. If a question consists of a series of choices in which there are p possibilities for the first choice, q possibilities for the second choice, r for the third, etc., then the number of ways in which the question can be done is just $p \times q \times r \times \ldots$ In other words, you just multiply together the number of ways each step can be done.

Example

How many BC licence plates for cars are there (barring reserved words, etc.)?

Answer: The patterns allowable are

letter-letter-letter number-number-number

number-number-number letter-letter-letter

(we'll ignore personalized plates or reserved words, etc.) Looking at the first pattern, there are 26 choices for each letter and 10 for each number. So we've got 26×26×26×10×10×10 = 17,576,000 plates.

The second pattern will have the same number, for a total of 35,152,000 plates using the two patterns.

Example

How many (250) area code phone numbers are there?

Answer: Phone numbers are of the form (250) ### – ####. To look at all possibilities, there are 10 digits for each #, so 10^7 or ten million possibilities.

Example

How many (250) area code phone numbers are there that don't start with zero?

Answer: Now we have only 9 choices for the first #, so 9×10^6 or nine million.

Example

How many (250) area code phone numbers are there that don't start with 911?

Answer: This one's more tricky. What we'll do is take the total number of phone numbers and subtract the number that **do** begin with 911. Numbers beginning with 911 will look like: (250) 911–####, so we'll have 10^4 choices.

Since there are 10,000,000 numbers in total, the number that don't start with 911 is (10,000,000 – 1000) = 9,999,000.

There's another rule that we should know regarding counting if we're using the word "or".

Example

How many numbers from 1 to 30 are a) divisible by 3? b) divisible by 5? c) divisible by 3 or 5?

Answer: Well, let's try the brute force method.

Divisible by 3: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30 total: 10

Divisible by 5: 5, 10, 15, 20, 25, 30 total: 6

Divisible by 3 or 5: number in first row (10) + number in second row (6) but we have to subtract 2, because otherwise we are counting 15 and 30 twice! So we get 10 + 6 – 2 = 14.

This brings up the addition rule:

$$n(A \text{ or } B) = n(A) + n(B) - n(AB)$$

Now, if the two situations don't have overlap, then n(AB) can be zero. We say then that the two situations are **mutually exclusive**.

Example

How many case-sensitive alpha-numeric passwords are there that have 6 or 7 characters?

Answer: First, look at what "case-sensitive, alpha-numeric" means. It means that capital letters are considered different from lowercase, so 52 letters instead of 26. Also, numbers are allowed, so 62 choices in total.

Number of 6-char passwords: $62\times62\times62\times62\times62\times62 = 62^6 = 5.68\times10^{10}$

Number of 7-char passwords: $62^7 = 3.52\times10^{12}$

Now, a password can either have 6 characters or 7 but not both, so to get the total, we just add $5.68\times10^{10} + 3.52\times10^{12} = 3.58\times10^{12}$.

Example

How many case-sensitive alpha-numeric passwords are there that have 6 characters and at least one number and letter?

Answer: This question is quite difficult as written. However, if we calculate instead the total number of passwords and subtract the number that don't have any numbers and the number that don't have any letters, we'll get the same result.

Number of 6-char passwords without any numbers: 52^6

Number of 6-char passwords without any letters: 10^6

So we get total = $62^6 - 52^6 - 10^6 = 3.70\times10^{10}$