Section 6.2: Combinations and Permutations

Factorials

If you've ever seen 10!, a number with an exclamation point behind it, it means that we take the following product:

$$10! = 10\times9\times8\times7\times \ldots 1 .$$

More generally, *n*! is called "*n*-factorial" and is the product of the number *n* and all of the positive integers less than or equal to n:

$$n! = n\times(n-1)\times(n-2)\times\ldots1 .$$

Example

Calculate $\frac{12!}{9!}$.

Answer: You probably have a factorial button on your calculator. But if you don't, you can shorten the calculation considerably by noting that

$$\begin{aligned}\frac{12!}{9!} &= \frac{12\times11\times10\times9\times8\times7\times\ldots1}{9\times8\times7\times\ldots1}\\ &= \frac{12\times11\times10}{1}\\ &= 1320\end{aligned}$$

Permutations

A permutation is an **ordered** group of objects chosen **without repetition** from a set of possibilities. If the number of objects we are choosing is *r* and the number of possibilities that we can choose from is *n* (with $n \geq r$), then this permutation is has the symbol $_nP_r$, where it can be calculated by

$$_nP_r = \frac{n!}{(n-r)!}$$

(Other commonly used symbols are P_r^n and $P(n,r)$.)

To see how this works (and to justify this strange equation), we should look at an example.

Example

How many four-digit PIN number for a bank card could you have if you are not allowed to repeat digits?

Answer: There are two ways to calculate this: you could say that there are 10 possibilities for the first digit, 9 for the second, 8 for the third, and 7 for the fourth (since you can't repeat numbers). Then your answer would be that the total number allowed would be $10\times9\times8\times7$, which equals 5040.

Alternatively, you could note that

$$10\times9\times8\times7=\frac{10\times9\times8\times7\times6\times5\times...1}{6\times5\times4\times...1}$$

so that you could say that $10\times9\times8\times7=\frac{10!}{6!}=\frac{10!}{(10-4)!}$, where 10 is the number of digits you are selecting from (n)and 4 is the number you are selecting (r).

So, ${}_{10}P_4=\frac{10!}{(10-4)!}$, and more generally ${}_nP_r=\frac{n!}{(n-r)!}$.

Combinations

A combination is an **unordered** group of objects chosen **without repetition** from a set of possibilities. If the number of objects we are choosing is r and the number of possibilities that we can choose from is n (with $n \geq r$), then this combination is has the symbol ${}_nC_r$, where it can be calculated by

$${}_nC_r=\frac{n!}{r!(n-r)!}$$

(Other commonly used symbols are C_r^n, $C(n,r)$, and $\binom{n}{r}$.)

In other words, ${}_nC_r$ can be calculated by taking the number of ordered arrangements ${}_nP_r$ and dividing by the number of ways you can arrange those r objects (which turns out to be $r!$).

Example

If you went to the library and found that there were seven books on the subject of bathtub races, and you decided to check out three of them, how many different selections could you possibly make?

Answer: You are choosing 3 from 7, so $r = 3$ and $n = 7$. Since the order that you check them out doesn't matter, we want ${}_nC_r$, so:

$$ {}_nC_r = \frac{n!}{r!(n-r)!} $$
$$ {}_7C_3 = \frac{7!}{3!4!} = 35 $$

An important difference between permutations and combinations is whether order matters. For example, order matters when listing the characters in computer passwords, student numbers, and words in general. Order does not matter when you are looking at a committee of people, the cards in a poker hand, or the winning numbers for the Lotto 6/49 game.

Example

Consider the set of letters A, B, C, D, and E. If you were to list all possible selections of two letters drawn from this set without repetition, how many choices would you have if a) order matters and b) order doesn't matter?

Answer:

Method #1: Well, there is a very small list here, so we could do this by brute force and list all of the possible choices:

	AB	AC	AD	AE
BA		BC	BD	BE
CA	CB		CD	CE
DA	DB	DC		DE
EA	EB	EC	ED	

Notice that I've left the diagonal blank because you can't repeat letters and so AA would not be allowed. Then if you count the number of choices, there are 20 of them. So the answer to the first part is that there are 20 possibilities.

To answer part b), you have to notice that AB and BA are considered to be the same when order doesn't matter. So all of the entries below the diagonal are repetitions of the ones above. We can then cross out all the ones below, leaving 10 possibilities when order doesn't matter.

Method #2: To do this using ${}_nP_r$ and ${}_nC_r$, we notice that part a) will be a permutation and b) will be a combination. For both of these, $n = 5$ (letters to choose from) and $r = 2$ (number of letters we are choosing). Then we get

a) ${}_nP_r = \dfrac{n!}{(n-r)!}$ so ${}_5P_2 = \dfrac{5!}{3!} = 20$

b) ${}_nC_r = \dfrac{n!}{r!(n-r)!}$ so ${}_5C_2 = \dfrac{5!}{2!3!} = 10$

Example

Pat has a test bank of 20 multiple-choice questions and 15 word problems. She wishes to create a quiz of ten multiple-choice questions and two word problems. How many different quizzes could she potentially make?

Answer: In general, a quiz is considered the same if it contains the same questions as another, so order does not matter. We should treat the two parts, multiple-choice and word-problem questions, separately:

combinations for multiple-choice $= {}_{20}C_{10} = \dfrac{n!}{r!(n-r)!} = \dfrac{20!}{10!10!} = 184,756$

combinations for word problems $= {}_{15}C_2 = \dfrac{n!}{r!(n-r)!} = \dfrac{15!}{2!13!} = 105$

so total # quizzes = (multiple-choice combos)×(word problem combos)

= 184,756×105 = 19,399,380

Example

Students are given a list of twelve computer games and asked to pick their three favourites. How many different lists could there be if

a) students just list their favourites in any order

b) students rank their favourite as #1, their second favourite as #2, etc.?

Answer:

In both of these questions, $r = 3$ and $n = 12$.

a) ${}_nC_r = \dfrac{n!}{r!(n-r)!}$ so ${}_{12}C_3 = \dfrac{12!}{3!9!} = 220$

b) ${}_nP_r = \dfrac{n!}{(n-r)!}$ so ${}_{12}P_3 = \dfrac{12!}{9!} = 1320$