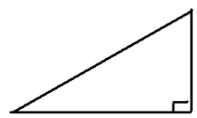
Section 4.1: Introduction to Trigonometry

Review of Triangles

Recall that the sum of all angles in any triangle is 180°. Let's look at what this means for a right triangle:

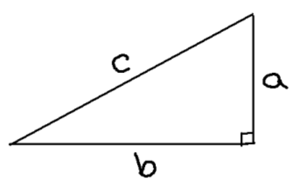


A right angle is an angle which measures 90°. A right triangle, then, has one 90° angle, and since the sum of all angles must be 180°, the other two angles in the triangle must add to 90°. In the diagram, the right angle is denoted by a little rectangle at the vertex (corner) of the right angle.

Since the other two angles sum to 90°, they must both be acute (less than 90°). Angles between 90° and 180° are called obtuse.

Pythagorean Theorem

The Pythagorean Theorem states that if a triangle is a right triangle, then $a^2 + b^2 = c^2$ (as shown in the diagram) and vice versa.

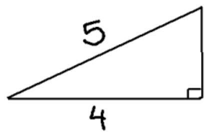


In the diagram we use the naming convention for right triangles, in which c is the hypotenuse (the side opposite the right angle) and a and b are the other two sides of the triangle in any order.

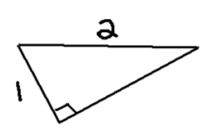
Example

Calculate the remaining side for each of the following triangles.

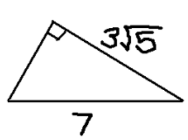
a)



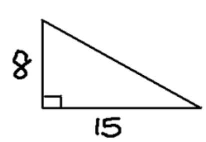
b)



c)



d)



Answer:

a) Let's call the remaining side a. Then $b = 4$, $c = 5$, and

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 4^2 &= 5^2 \\ a^2 + 16 &= 25 \\ a^2 &= 9 \\ a &= 3 \end{aligned}$$

To be rigorous, we really should say that a could be ±3, but triangles can't have negative sides, so we only take the positive answer, 3.

b) Let's call the remaining side b. Then $a = 1$, $c = 2$, and

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 1^2 + b^2 &= 2^2 \\ 1 + b^2 &= 4 \\ b^2 &= 3 \\ b &= \sqrt{3} \end{aligned}$$

c) Let's call the remaining side a. Then $b = 3\sqrt{5}$, $c = 7$, and

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + \left(3\sqrt{5}\right)^2 &= 7^2 \\ a^2 + 45 &= 49 \\ a^2 &= 4 \\ a &= 2 \end{aligned}$$

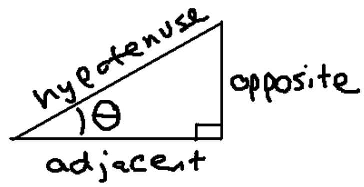
(Remember that $\left(3\sqrt{5}\right)^2 = \left(3\sqrt{5}\right)\left(3\sqrt{5}\right) = 3^2\left(\sqrt{5}\right)^2 = 9 \times 5 = 45$.)

d) The remaining side is the hypotenuse, c. Then $a = 8$, $b = 15$, and

$$\begin{aligned} c^2 &= a^2 + b^2 \\ c^2 &= 8^2 + 15^2 = 64 + 225 = 289 \\ c &= 17 \end{aligned}$$

Trigonometric Ratios

Let's consider a right triangle as shown in the diagram below. We've labeled one of the two acute angles with the variable name θ, which is the Greek letter "theta". There is a naming convention, then, for the remaining two sides of the triangle. The longest side is still called the hypotenuse, but now the side opposite to the angle θ is called the "opposite", while the side next to the θ is called the "adjacent" side.



The trig ratios are then just ratios of these sides. The three basic trig functions, which are the ones we will study in this course, are called **sine**, **cosine**, and **tangent**. They are defined as follows.

$$\sin\theta = \frac{opposite}{hypotenuse}$$

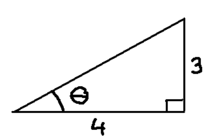
$$\cos\theta = \frac{adjacent}{hypotenuse}$$

$$\tan\theta = \frac{opposite}{adjacent}$$

The easy way to remember them is to take the first letter of each word: SOHCAHTOA. Then the SOH stands for Sin/Opp/Hyp, CAH for Cos/Adj/Hyp, and TOA for Tan/Opp/Adj.

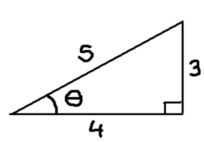
Example

Calculate the three basic trig functions of θ exactly for the following triangle.



Answer:

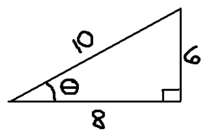
First, we need to calculate the hypotenuse. We can either note that it's the same 3-4-5 triangle as in the first example, or use the Pythagorean theorem to calculate that $c = 5$.



Then we note that for this θ, the opposite side is 3 and the adjacent side is 4. So,

$$\sin\theta = \frac{opp}{hyp} = \frac{3}{5}$$
$$\cos\theta = \frac{adj}{hyp} = \frac{4}{5}$$
$$\tan\theta = \frac{opp}{adj} = \frac{3}{4}$$

What happens if we look at the following triangle?



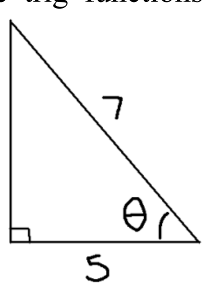
Since this is just the 3-4-5 triangle scaled up or enlarged by a factor of 2, the two triangles are similar, so the two θs are equal to each other. Let's look at what the trig ratios would be for the θ in the larger triangle.

$$\sin\theta = \frac{opp}{hyp} = \frac{6}{10} = \frac{3}{5}$$
$$\cos\theta = \frac{adj}{hyp} = \frac{8}{10} = \frac{4}{5}$$
$$\tan\theta = \frac{opp}{adj} = \frac{6}{8} = \frac{3}{4}$$

And you'll notice that the trig ratios are the same as for the previous triangle. What that means is that the **trig ratios are only a function of the angle θ** and do not depend on the size of the triangle. After all, if you scale the triangle up by 2, then all three sides increase by the same factor. When you take the ratio, that scale factor will be in both the numerator and denominator of the fraction, so will cancel.

Example

Calculate the three basic trig functions of θ exactly for the following triangle.



Answer:

First, we need to calculate the remaining side, using the Pythagorean theorem. Let's call it *a*. Then

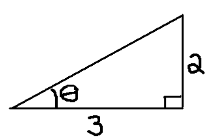
$$a^2 + b^2 = c^2$$
$$a^2 + 5^2 = 7^2$$
$$a^2 + 25 = 49$$
$$a^2 = 24$$
$$a = \sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}$$

Then we note that for this θ, the opposite side is $2\sqrt{6}$ and the adjacent side is 5. Therefore,

$$\sin\theta = \frac{opp}{hyp} = \frac{2\sqrt{6}}{7}$$
$$\cos\theta = \frac{adj}{hyp} = \frac{5}{7}$$
$$\tan\theta = \frac{opp}{adj} = \frac{2\sqrt{6}}{5}$$

Example

Calculate the three basic trig functions of θ exactly for the following triangle.



Answer:

First, we need to calculate the hypotenuse, using the Pythagorean theorem.

$$c^2 = a^2 + b^2$$
$$c^2 = 2^2 + 3^2 = 4 + 9 = 13$$
$$c = \sqrt{13}$$

Then we note that for this θ, the opposite side is 2 and the adjacent side is 3. So,

$$\sin\theta = \frac{opp}{hyp} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

$$\cos\theta = \frac{adj}{hyp} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\tan\theta = \frac{opp}{adj} = \frac{2}{3}$$

Remember that you have to rationalize the denominator when you are simplifying fractions containing radicals!