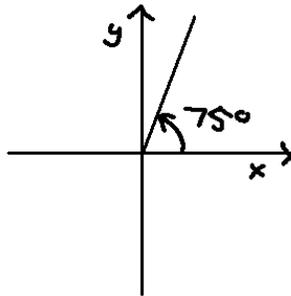


Section 4.4: Trig Functions of Any Angle

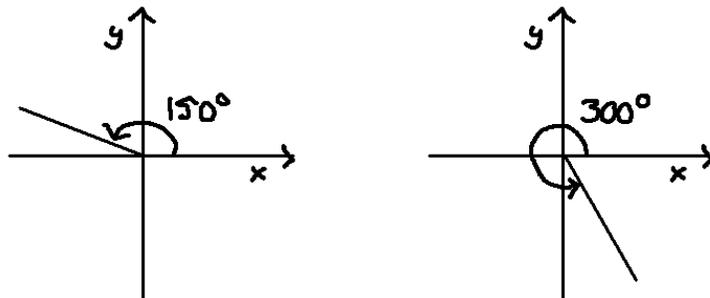
Angles in Standard Position

Suppose we don't want to restrict ourselves to acute angles. Let's take a look at how we can define larger angles using a coordinate plane.

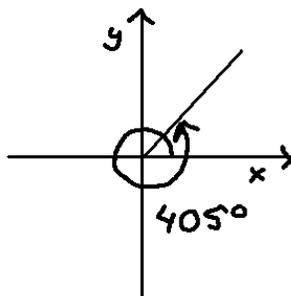
An **angle in standard position** is an angle whose initial arm is on the positive x -axis and which rotates counter-clockwise to the final or **terminal** arm. For example, a 75° angle would be plotted as follows.



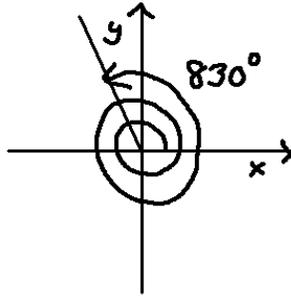
But there's no reason to restrict our angles to the first quadrant. For angles larger than 90° , we can simply keep rotating until we reach the terminal arm. To see what that looks like, let's look at the 150° and 300° angles graphed below.



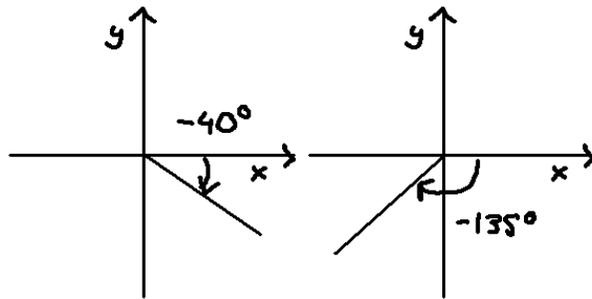
In fact, we don't have to restrict ourselves to angles less than 360° . We can just keep going, so that 405° would continue for another 45° past the x -axis, as in the diagram.



Similarly, 830° would be two full rotations (720°) plus another 110° , putting the terminal arm in Quadrant II.

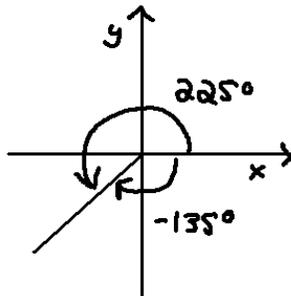


Then negative angles can be represented by **clockwise** rotations. -40° and -135° are plotted below, and there is no reason that you couldn't also have -900° and other angles requiring more than one full rotation.



Coterminal Angles

If you take a look at the diagram below, you'll notice that 225° and -135° both have the same terminal arm. Angles with the same terminal arm are called **coterminal angles**.



If you think about it, though, there's no reason that you couldn't just add another rotation to 225° to get $(225^\circ + 360^\circ) = 585^\circ$ as another coterminal angle. In fact, there are an infinite number of coterminal angles for 225° , and one way to represent this would be to say that if θ is coterminal with 225° , then

$$\theta = 225^\circ + n360^\circ$$

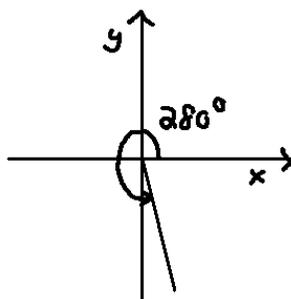
where n is an integer (if n is negative, then you are just adding counterclockwise rotations).

Example

Plot 280° in the coordinate plane, and list one positive and one negative coterminal angle.

Answer

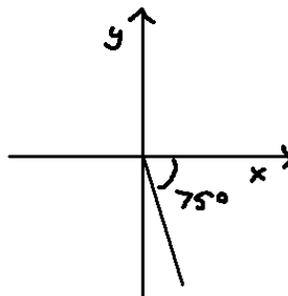
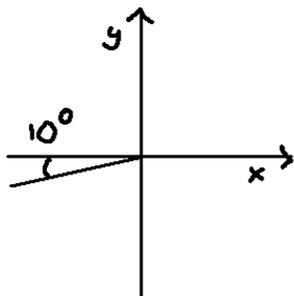
The sketch of 280° is shown below. The angle between the terminal arm and the x -axis is 80° , so -80° is a negative coterminal angle. Adding one rotation to 280° gives $(280^\circ + 360^\circ = 640^\circ)$ as a positive coterminal angle.



However, you don't just have to limit yourself to one rotation either way. Other acceptable answers would include $(280^\circ + 5 \times 360^\circ = 2080^\circ)$, $(280^\circ + (-10) \times 360^\circ = -3320^\circ)$ and similar answers. In fact, each angle has an infinite number of coterminal angles.

Reference Angles

When θ is in standard position, its **reference angle** is the angle between the terminal arm and the nearest x -axis. Reference angles are always positive. For example, the angle 190° has a reference angle of 10° and the angle 285° has a reference angle of 75° , as shown below.

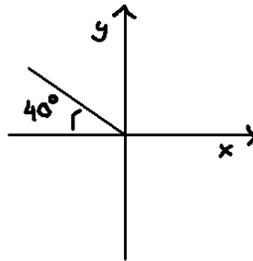


Example

What is the reference angle to -580° ?

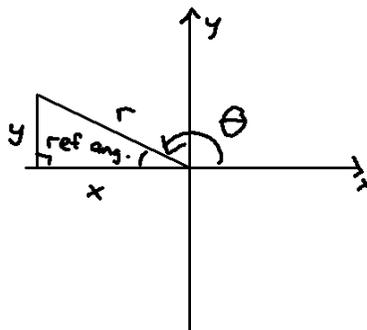
Answer

-580° is one clockwise rotation plus 220° , so will have its terminal arm in QII, as shown below. The reference angle is then the angle between the terminal arm and the negative x -axis, 40° .

**Generalizing from Triangles to Coordinate Plane**

When an angle is in standard position, then it can take on any value in the set of real numbers. What does this mean for trig functions of larger angles?

Consider the angle 150° . It lies in the second quadrant, as shown below, and has a reference angle of 30° . If we drop a line down to the closest x -axis to make a right angle, then we have set up a little right triangle containing the reference angle. Since we are now in a coordinate system, we can consider the sides of the triangle to now be **coordinates** instead of lengths.



If we label the adjacent side as x and the opposite side as y , then the setup looks very similar to our right-triangle trigonometry of the previous section. However, the x - and y -coordinates could take on **positive** values, unlike lengths of sides which are always positive. As you can see in the diagram above, reference angles in QII will give triangles in which the x -value is negative while the y -value is positive.

We've labeled the hypotenuse as r in the diagram. It is considered to be the distance from the point (x,y) to the origin, and is always positive. We can still use the Pythagorean theorem to calculate its value from x and y .

Looking at the diagram once again, then the sine of any angle θ in standard position will be

$$\sin \theta = \frac{\textit{opposite}}{\textit{hypotenuse}} = \frac{y}{r}$$

$$\cos \theta = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{x}{r}$$

$$\tan \theta = \frac{\textit{opposite}}{\textit{adjacent}} = \frac{y}{x}$$

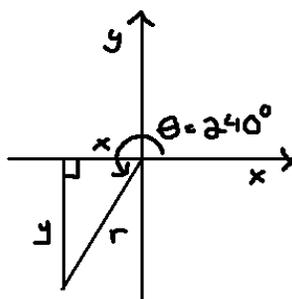
where x and y can now be any real number and r is a positive quantity. The resulting trig functions can then take on both positive and negative values.

Example

Will the trig functions of 240° be positive or negative?

Answer

240° is in the third quadrant, as shown below. Therefore, both x and y will be negative, making $\sin \theta$ and $\cos \theta$ negative as well. However, $\tan \theta$ will be positive, since y/x will be a positive quantity.



In fact, if you calculate the values of the trig functions of 240° using a calculator, you'll find that

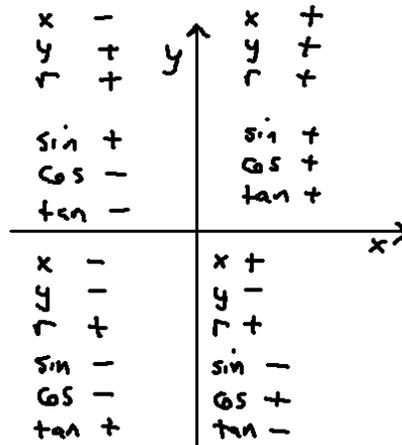
$$\sin 240^\circ = -0.866$$

$$\cos 240^\circ = -0.5$$

$$\tan 240^\circ = 1.732$$

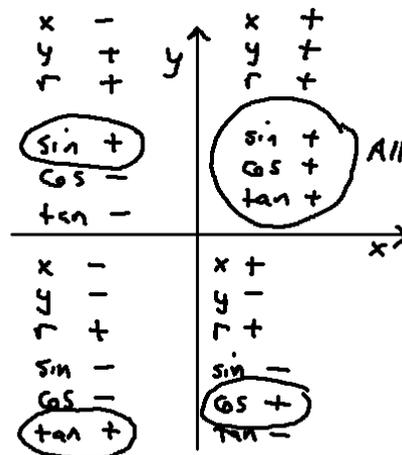
and the sign of each trig function (positive or negative) is consistent with our analysis above.

Is there a quick way to determine whether the trig functions for a particular angle will be positive or negative? Consider the following chart. In each quadrant, the sign for x -values and y -values is written in. Then the sign of each trig function is determined.

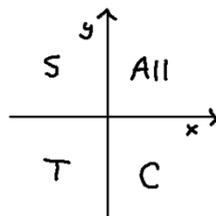


For example, in Quadrant IV, x -values will be positive and y -values will be negative, giving rise to a positive cosine (x/r) and negative sine (y/r) and tangent (y/x).

An easy way to remember this is to circle the trig functions in each quadrant that are positive:



Then All trig functions in Quadrant I are positive, Sine is positive in Quadrant II, Tangent in QIII, and Cosine in Quadrant IV, like so:



And this can be remembered by the phrase, “All Students Take Calculus”, starting with the All in quadrant I and rotating counterclockwise, just as in angles. Other similar sayings include “All Suckers Take Chemistry”, “All Saskatchewaners Tip Cows”, or anything that uses ASTC that is easy to remember.

Example

State whether the three basic trig functions are positive or negative for angles in Quadrant IV.

Answer

From the “All Students Take Calculus” rule, Quadrant IV has C for cosine positive, so the sine and tangent functions will be negative there.

Example

Calculate the values of the following trig functions using a calculator. Round your answers to four decimal places.

a) $\sin 2543^\circ$

b) $\cos -172^\circ$

c) $\tan 270^\circ$

d) $\sin 540^\circ$

Answer:

a) $\sin 2543^\circ = 0.3907$

b) $\cos -172.224^\circ = -0.9908$

c) $\tan 270^\circ = \text{undefined}$

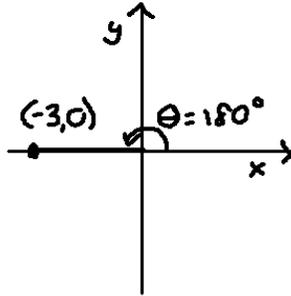
d) $\sin 540^\circ = 0$ (exact, but you can add decimal places if you wish)

Quadrantal Angles

What if the angle you are interested lies on one of the axes? What happens to the trig functions of that angle?

The answer lies in using the system of coordinates we've developed. Even if you cannot draw a right triangle if the terminal arm is on an axis, you may still take a point on that line and calculate x , y , and r .

For instance, suppose we are interested in calculating the three basic trig functions for $\theta = 180^\circ$. Let's look at the diagram below, and pick a point somewhere on the terminal arm, such as $(-3, 0)$.



For any point we pick, the first coordinate must be a negative real number, since it lies on the negative x -axis, and the second coordinate must be zero since $y = 0$. For our particular point, r is $+3$, since the distance from the origin to the point we've chosen is 3 .

Consequently,

$$\sin \theta = \frac{y}{r} = \frac{0}{3} = 0$$

$$\cos \theta = \frac{x}{r} = \frac{-3}{3} = -1$$

$$\tan \theta = \frac{y}{x} = \frac{0}{-3} = 0$$

It doesn't make sense to talk about the opposite, adjacent, and hypotenuse when there is no right triangle associated with the calculation, but the coordinates can still be placed in the appropriate places to calculate the trig functions, as we've done before.

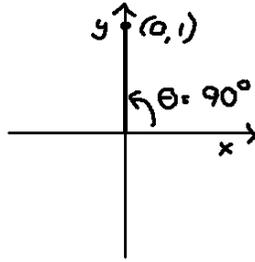
Even if we changed the coordinates of our point to something else, note that r will always just be the absolute value of the x -coordinate, since it's the distance, and even if we look at the point $(-7, 0)$ or more generally $(x, 0)$ where x is negative, we'll still get the sine and tangent of the angle to equal zero and the cosine will equal negative one, because we'll be dividing the x value by the absolute value of x .

Example

Calculate the three basic trig functions of 90° using coordinates.

Answer

The terminal arm of $\theta = 90^\circ$ lies on the $+y$ -axis, as shown below.



We need to choose a point on the +y-axis. Let's choose (0, 1) for convenience. Then $r = 1$ (distance from origin to our point), and

$$\sin \theta = \frac{y}{r} = \frac{1}{1} = 1$$

$$\cos \theta = \frac{x}{r} = \frac{0}{1} = 0$$

$$\tan \theta = \frac{y}{x} = \frac{1}{0} = \text{undefined}$$

As you can see from the previous examples, the three basic trig functions of the quadrantal angles take on four possible values: +1, 0, -1, and "undefined". And, yes, it's quite possible for the tangent of an angle to be undefined. In fact, $\tan 90^\circ$ and $\tan 270^\circ$ and the tangent of any angle coterminal to these two will be undefined, since the x -coordinate must be zero.