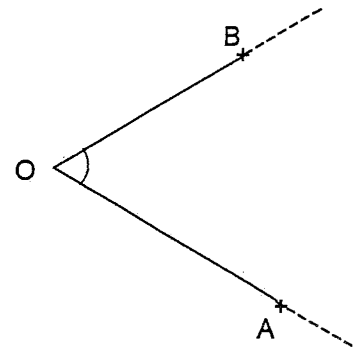
Section 1. Triangles

An **angle** is formed by two rays from a point O, say, OA and OB. The rays OA and OB are called the sides of the angle and O is called the vertex of the angle. The angle illustrated is called the angle AOB (or angle BOA). The symbol $\angle$ stands for the word angle.

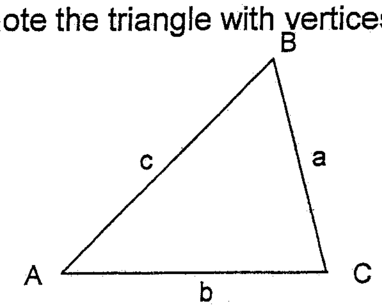


The unit used in measurement of angle is degree. There are 360 degrees in a complete revolution. For more accurate measurement, each degree is divided into 60 parts, called minutes, and each minute is divided into 60 parts, called seconds. Usually " ° ", " ' " and " " " are used to denote degrees, minutes and seconds respectively. For example, if $\angle$AOB is an angle of 34 degrees 56 minutes and 20 seconds, we write $\angle AOB = 34°56'20''$.

(1) An angle of 90° is called a **right** angle.
(2) An angle of 180° is called a **straight** angle.
(3) An angle between 0° and 90° is called an **acute** angle.
(4) An angle between 90° and 180° is called an **obtuse** angle.

If three distinct points A, B and C are not collinear, then we can construct a triangle from the 3 points. The points A, B and C are called the vertices (plural of vertex) of the triangle; the line segment AB (or BA), BC (or CB) and CA (or AC) are the sides of the triangle, and the angles $\angle$BAC, $\angle$CBA and $\angle$ACB are called its interior angles (or just its angles for short). The line segments BC, CA and AB are frequently denoted by a, b and c, respectively.

The symbol ΔABC is used to denote the triangle with vertices A, B and C.



Theorem 1.1 **The sum of the interior angles of a triangle is equal to 180° (two right angles).**

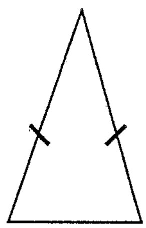
Triangles may be classified in several ways.

I. Classification of triangles using side lengths:

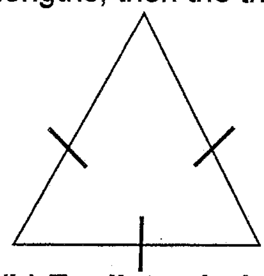
(1) If two sides of a triangle are of equal length, then the triangle is **isosceles**.

(2) If all three sides of a triangle are of the same length, then the triangle is **equilateral**. An equilateral triangle is also isosceles.

(3) If all three sides of a triangle are of different lengths, then the triangle is **scalene**.



(a) Isosceles triangle



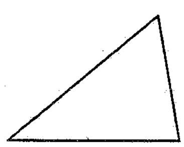
(b) Equilateral triangle

II. Classification of triangles using the sizes of angles:

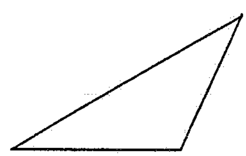
(1) If all three angles of a triangle are acute, then the triangle is called **acute**.

(2) If one of the angles of a triangle is obtuse, then the triangle is called **obtuse**.

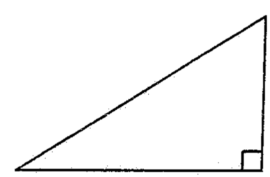
(3) If one of the angles of a triangle is a right angle, then the triangle is a **right triangle** or a **right-angled triangle**.



(a) Acute triangle



(b) Obtuse triangle



(c) Right triangle

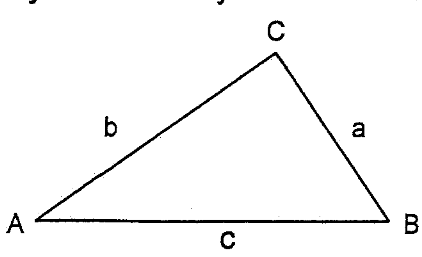
Two angles that add up to 90° are called **complementary angles**.

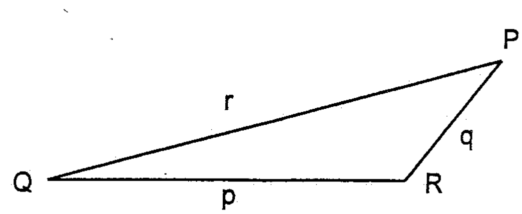
Theorem 1.2 **The two acute interior angles of a right triangle are complementary.**

Property 1.3 **The measures of the three sides and the measures of the three corresponding angles of a triangle are in the same order (i.e. largest angle opposite the longest side, smallest angle opposite the shortest side).**

$\angle A < \angle B < \angle C \leftrightarrow a < b < c$; $\angle Q < \angle P < \angle R \leftrightarrow q < p < r$

(The symbol "$x \leftrightarrow y$" stands for a correspondence between the two statements x and y.)





Another useful property that relates the three lengths of the sides of any triangle is

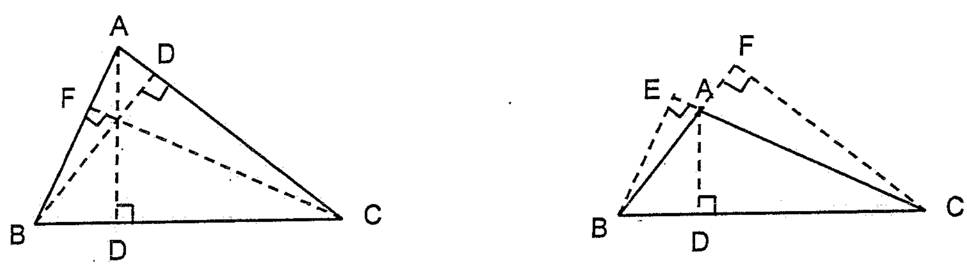
Property 1.4 (The Triangle Inequality Theorem)
The sum of any two sides of a triangle is greater than or equal to the remaining side. Equality holds if and only if the three vertices of the triangle are collinear (and the triangle degenerates into a straight line with zero area).

Corollary 1.5 **For a triangle with sides of lengths a and b, the length of the remaining side x satisfies the inequality:**

$$|a - b| < x < a + b.$$

Example 1.6 In ΔABC, if a = 10, b = 7, what is the possible range of c? .

For a triangle ΔABC, relative to any of the three sides as the **base** (which is a line segment connecting two of the three vertices of the triangle) the **height** (or **altitude**) of the triangle is the length of the perpendicular line segment joining the remaining vertex to this base.

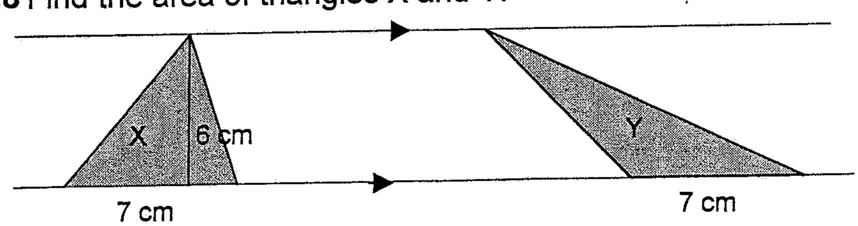


Base	AB	BC	AC
Height (or Altitude)	CF	AD	BE

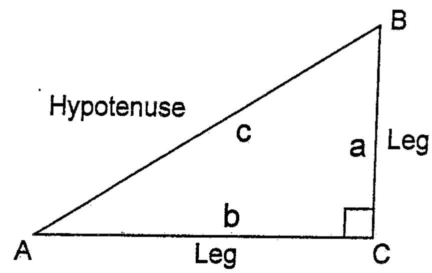
When there is no possible confusion from the content, we sometimes refer the length of a base to be the base.

Theorem 1.7 **The area of a triangle is half of the product of the base and the height.**

Example 1.8 Find the area of triangles X and Y.



Let ΔABC be a right triangle with the right angle at the vertex C. The side opposite the right angle is the **hypotenuse** and the two sides adjacent to the right angle are the **legs**. The Pythagorean theorem relates the three sides of a right triangle.

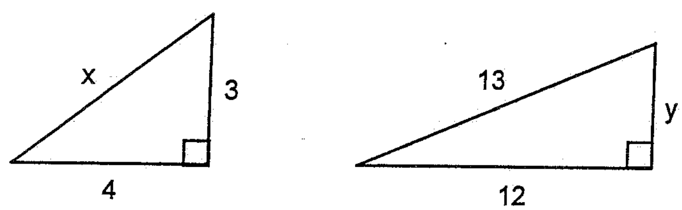


Theorem 1.9 **(Pythagorean Theorem)**
The sum of the squares of the lengths of the two legs of a right triangle is equal to the square of the length of the hypotenuse.

$$a^2 + b^2 = c^2$$

(Remember that in this formula, c is the length of the hypotenuse, and a and b are the lengths of the two legs.)

Example 1.10
(a) Find x in figure (a).
(b) Find y in figure (b).



(a) **(b)**

Theorem 1.11 **(Converse of Pythagorean Theorem)**
If the lengths of the three sides of a ΔABC satisfy the condition $a^2 + b^2 = c^2$, then ΔABC is a right triangle with the right angle at the vertex C.

Example 1.12 Which of these lengths of the sides of a right triangle?
(a) 9, 12, 15 (b) 4, 5, 6 (c) 10, 24, 26

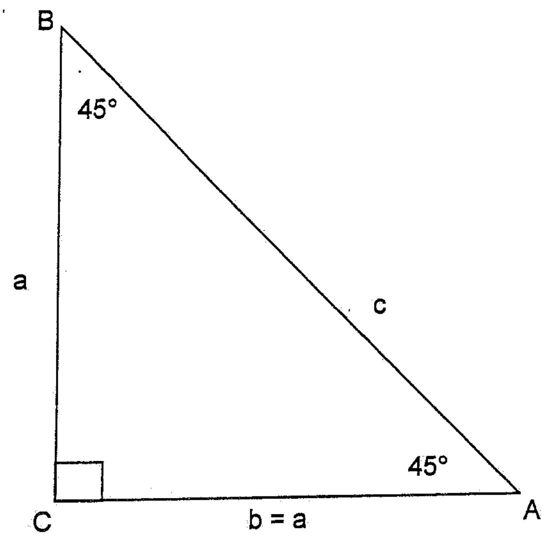
Applications (Special Angles)
Given the lengths of any two sides of a right triangle, we can solve for the length of the remaining side with the Pythagorean theorem.
In general, the three sides of a right triangle are related to the interior acute angles through trigonometric functions. However, there are two special right triangles of which we know the exact ratio of their three sides. These are the right triangles with interior acute angles 45° and 45°, and 30° and 60°.

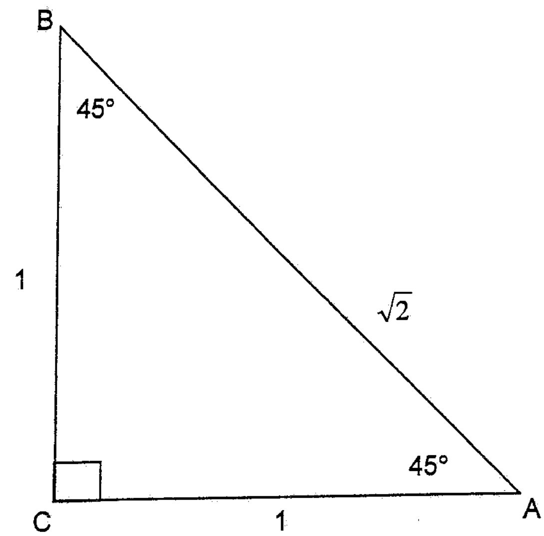
45°-45°-90° Right Triangle

Since the two interior acute angles of the ΔABC are equal, by property 1.3, ΔABC is an isosceles triangle with AC equal to BC, i.e. b = a; applying the Pythagorean theorem yields the result:

$$a^2 + a^2 = c^2 \Rightarrow 2a^2 = c^2 \Rightarrow \frac{c^2}{a^2} = \frac{2}{1} \Rightarrow \frac{c}{a} = \frac{\sqrt{2}}{1}$$

Hence $a:b:c = 1:1:\sqrt{2}.$



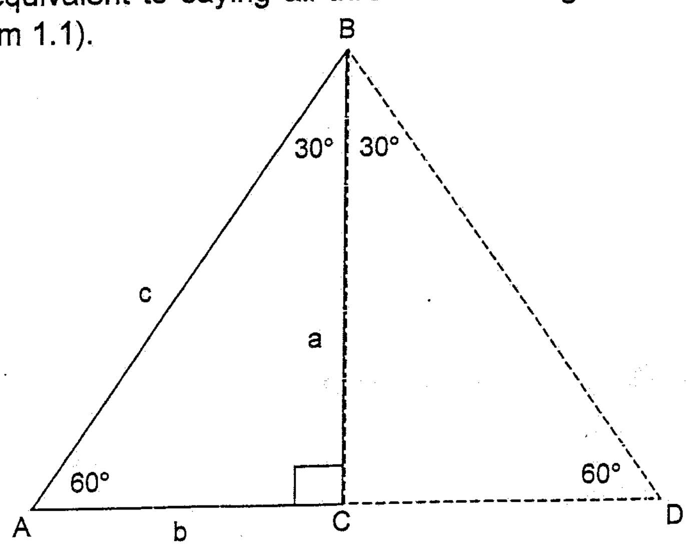


Note: The actual lengths of the sides of a 45°-45°-90° triangle are multiples of the numbers we derived here.

30°-60°-90° Right Triangle

For a right triangle with interior acute angles 30° and 60°, we can create an equilateral triangle by attaching another copy of the right triangle to itself as shown in the figure below.

Note: We defined an equilateral triangle as a triangle with all three sides of equal length. By property 1.3, this is equivalent to saying all three interior angles are equal, each being 60° (by theorem 1.1).

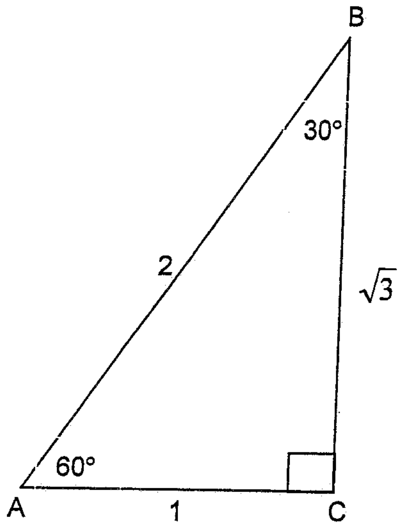


AC = ½ (AD) = ½ (AB) ⇒ b = ½ (c)
Applying the Pythagorean theorem,

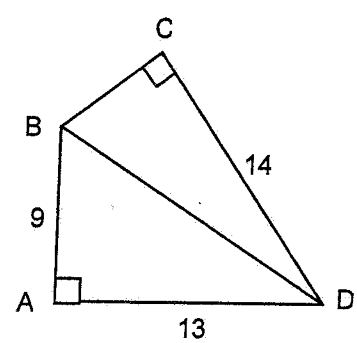
$$a^2+\left(\frac{c}{2}\right)^2=c^2 \Rightarrow a^2+\frac{1}{4}c^2=c^2 \Rightarrow a^2=\frac{3}{4}c^2 \Rightarrow a=\frac{\sqrt{3}}{2}c$$

In other words, $a:b:c=\frac{\sqrt{3}}{2}:\frac{1}{2}:1$, or equivalently $a:b:c=\sqrt{3}:1:2$

Note: The actual lengths of the sides of a 30°-60°-90° triangle are multiples of the numbers we derived here.



Example 1.13 Find (a) BD and (b) BC.



Example 1.14 Two boats A and B leave a port P at the same time. Boat A sails north-east at a speed of 33 km/h and B sails south-east at a speed of 40 km/h. Find the distance between the two boats 2 hours later.

Example 1.15 The coordinates of points A and B are (3,2) and (5,7), respectively. Find the length of AB.

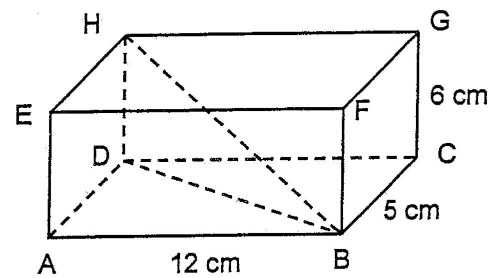
Example 1.15 can easily be generalized into the following distance theorem.

Theorem 1.16 **(Distance Formula)**
The distance AB between points A = (x_1, y_1) and B = (x_2, y_2) is given by

$$AB = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

Example 1.17 ABCDEFGH is a rectangular box. Find the length of

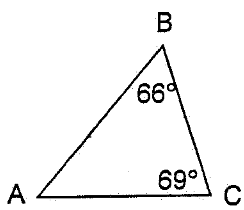
(a) DB and (b) HB.



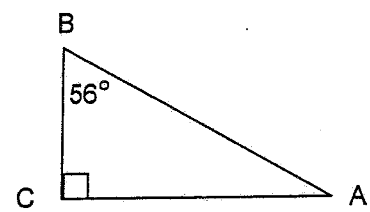
Exercise 1

Find ∠BAC in Question 1 and 2.

1.

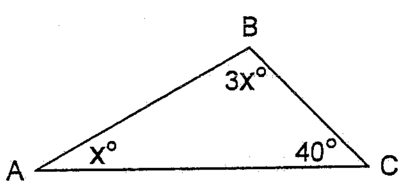


2.

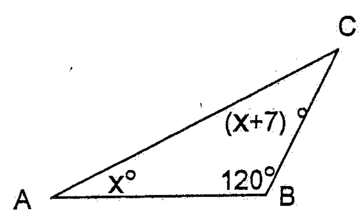


Find the value of x in Question 3 and 4.

3.

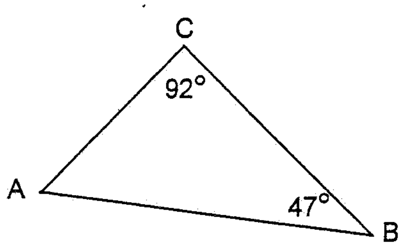


4.

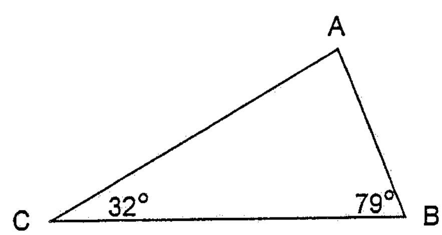


Find the size of ∠A in each of the following figures.

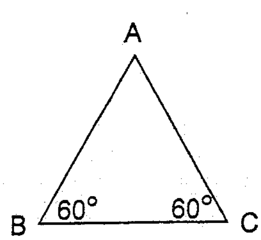
5.



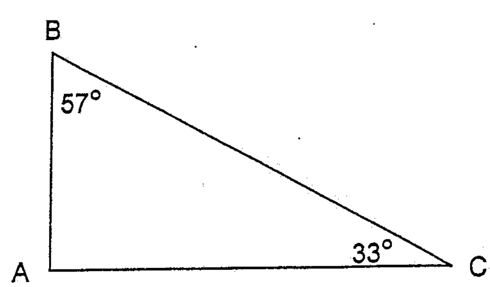
6.



7.



8.

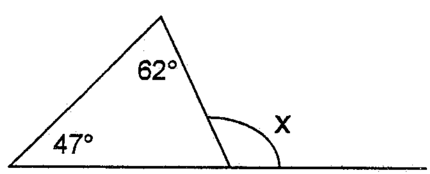


9. Classify the triangle in Question 1 using the sizes of angles.
10. Classify the triangle in Question 2 using the sizes of angles.
11. Classify the triangle in Question 3 using the sizes of angles.
12. Classify the triangle in Question 4 using the sizes of angles.
13. Classify the triangle in Question 5 using the sizes of angles.
14. Classify the triangle in Question 6 using the sizes of angles.
15. Classify the triangle in Question 7 using the sizes of angles.
16. Classify the triangle in Question 8 using the sizes of angles.

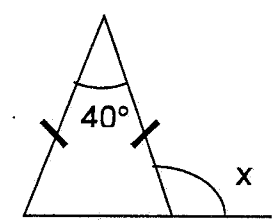
17. In ΔABC, if $\angle A = \angle B = 40°$, find $\angle C$.
18. In ΔABC, if $\angle A = \angle B$ and $\angle C = 40°$, find $\angle A$.
19. In ΔABC, if $\angle A = \angle B = \angle C$, find $\angle A$.
20. In ΔABC, if $\angle A = 2\angle B$ and $\angle C = 3\angle B$, find $\angle A$, $\angle B$ and $\angle C$.
21. In ΔABC, if $\angle A = 2\angle B$ and $\angle C = 3\angle A$, find $\angle A$, $\angle B$ and $\angle C$.
22. In ΔABC, if $\angle A = \angle B + 10°$ and $\angle C = \angle A + 10°$, find $\angle A$, $\angle B$ and $\angle C$.
23. Is it possible for a triangle to have two right angles? Explain briefly.
24. Is it possible for a triangle to have two obtuse angles? Explain briefly.
25. In ΔABC, $a = 10$, $b = 7$. What is the possible range of c?
26. In ΔABC, $a = 12$, $c = 5$. What is the possible range of b?
27. In ΔABC, $a = 3$, $b = 3$. What is the possible range of c?

Find the unknown marked angles in Questions 28 – 34.

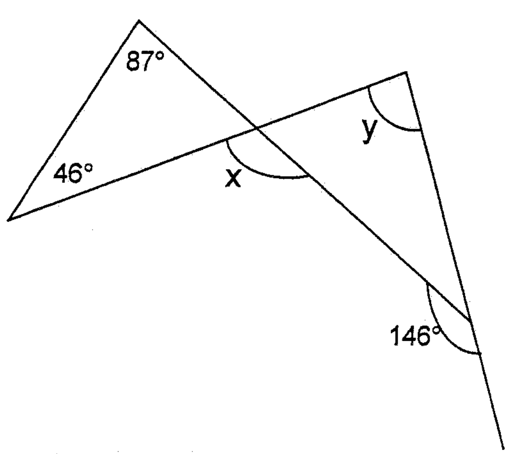
28.



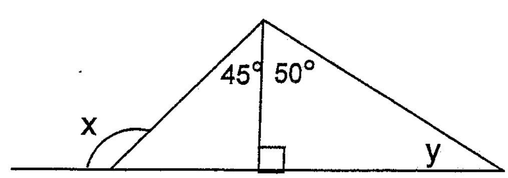
29.



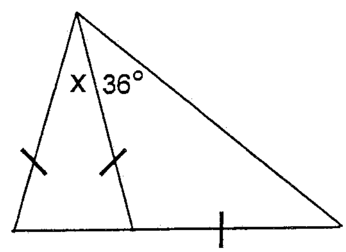
30.



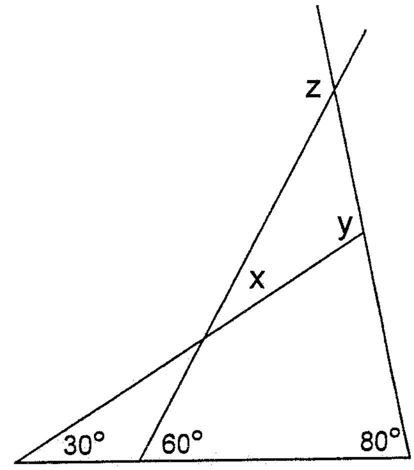
31.



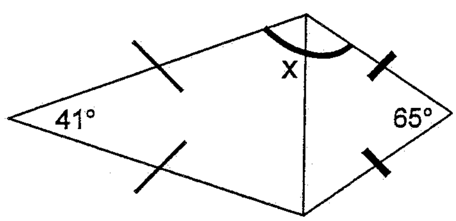
32.



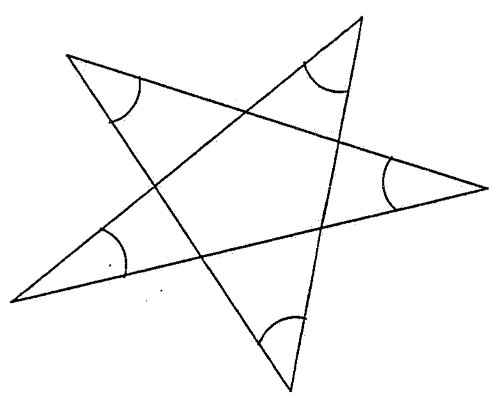
33.



34.



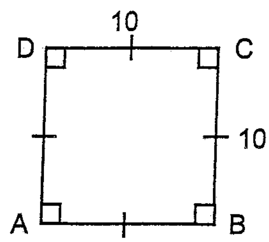
35. Find the sum of all vertex angles of a star.

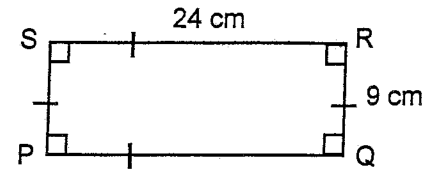


36. Find the value (to 2 decimal places) of the unknown variable:

(a) $x^2 = 9^2 + 16^2$
(b) $a^2 = 7^2 + 10^2$
(c) $35^2 = x^2 + 8^2$
(d) $13^2 = 4^2 + y^2$
(e) $26^2 = z^2 + 10^2$

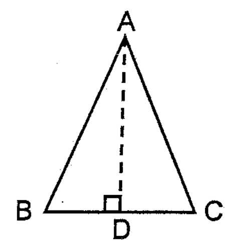
37. Find the length of the diagonal AC of the square ABCD.



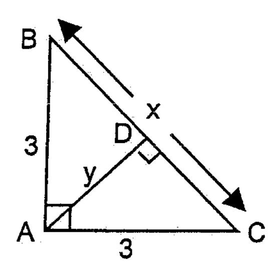


38. Find the length of the diagonal PR of the rectangle PQRS.

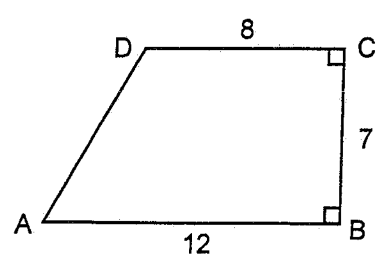
39. ΔABC is an isosceles triangle with AB = AC = 13 cm, and base BC = 10 cm. Find the length of the altitude AD.
(Hint: The altitude AD bisects BC.)



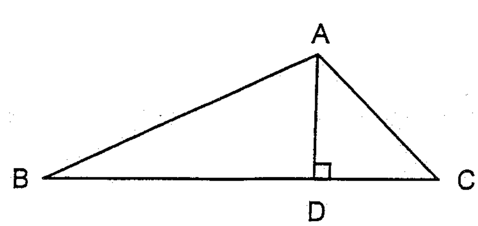
40. Find the values of x and y.
(Hint: The altitude AD bisects BC.)



41. Find AC and BD.

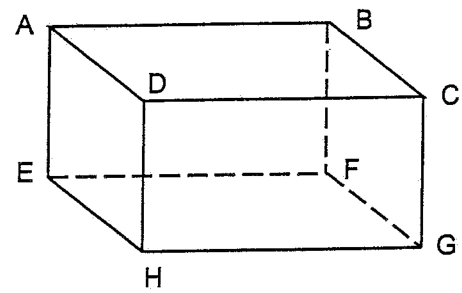


42. AB = 13 cm, BD = 12 cm and AC = 8 cm. Find AD and DC.



43. In the figure for the previous question, AB = 26 cm, AD = 10 cm and BC = 34 cm. Find BD and AC.

44. ABCDEFGH is a rectangular box where AB = 12 cm, AD = 9 cm and AE = 8 cm. Find (a) BD, (b) BH.



45. In the figure for the previous question, ABCDEFGH is a rectangular box where AB = 12 cm, BD = 13 cm and DH = 12 cm. Find (a) AD, (b) BH.

In Questions 46 - 55, find the distance between A and B.

46. A = (3, 7), B = (6, 9)
47. A = (-2, 7), B = (2, 9)
48. A = (-2, -4), B = (16, 9)
49. A = (0, 0), B = (6, 8)
50. A = (0, 0), B = (12, 5)
51. A = (0, 0), B = (12, -5)
52. A = (0, 0), B = (-12, 5)
53. A = (0, 0), B = (-12, -5)
54. A = (0, 0), B = (5, 12)
55. A = (3, 7), B = (15, 12)