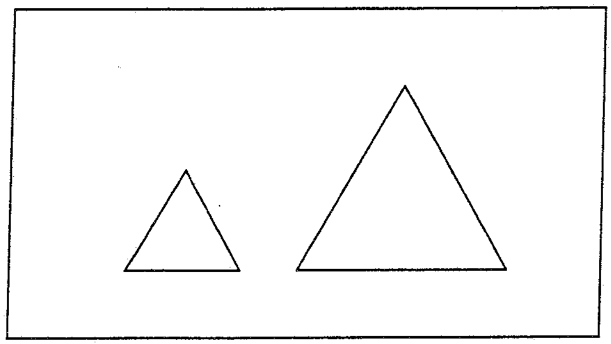
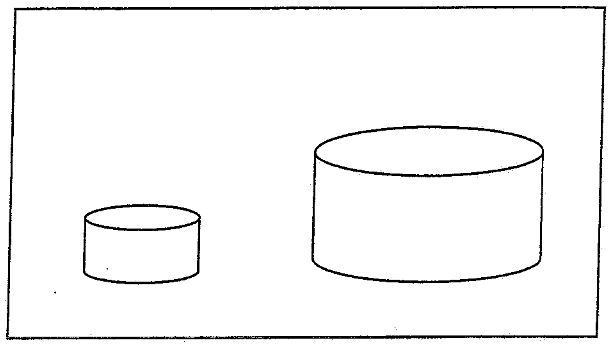
Section 2. Similar Triangles

Idea of Similar Figures

Consider the pairs of objects in the following figures.





Except for the size, the figures of each pair look the same. In plane geometry, two figures are said to be similar if they have the same shape. In particular, two triangles are similar if the angles of one are the same as the angles of the other. This is illustrated in first figure above. The triangle on the right is an enlargement of the triangle on its left, and the triangle on the left is a contraction of the triangle on its right.

Proportion Theorem for Triangles

We express the similarity relationship between ΔABC and $\Delta A'B'C'$ by writing

$\Delta ABC \sim \Delta A'B'C'$ (read "ΔABC is similar to $\Delta A'B'C'$ ").

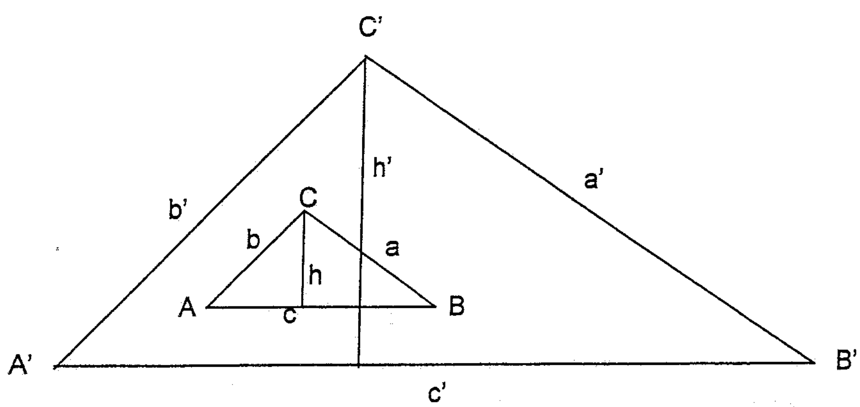
It can be proved that if $\Delta ABC \sim \Delta A'B'C'$ then $\Delta A'B'C'$ is an enlarged (reduced) copy of ΔABC, in the sense that, there is $k \geq 1$ $(0 < k \leq 1)$ such that

$$a' = ka, \quad b' = kb, \quad c' = kc$$

In the figure below, $\Delta A'B'C'$ is an enlarged copy of ΔABC by factor of $k > 1$.

Note that $a' = ka$, $b' = kb$, $c' = kc$ and $h' = kh$

which implies $k = \dfrac{a'}{a} = \dfrac{b'}{b} = \dfrac{c'}{c} = \dfrac{h'}{h}$.



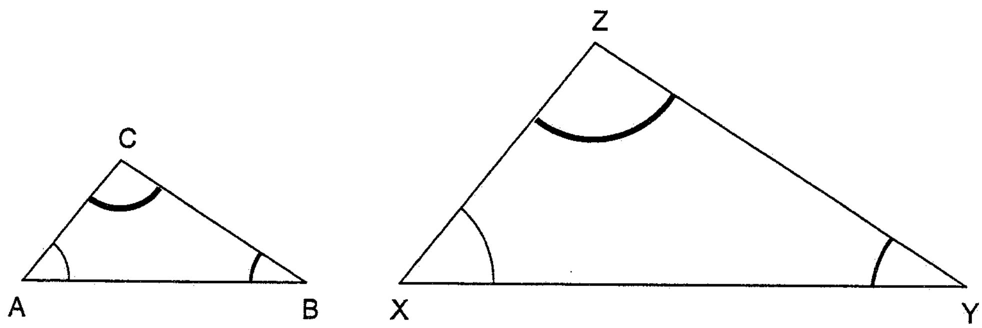
Theorem 2.1 **Ratios of corresponding parts of similar triangles are equal.**

Notice that two figures do not have to be of the same size in order to be similar.

Similarity Theorems for Triangles

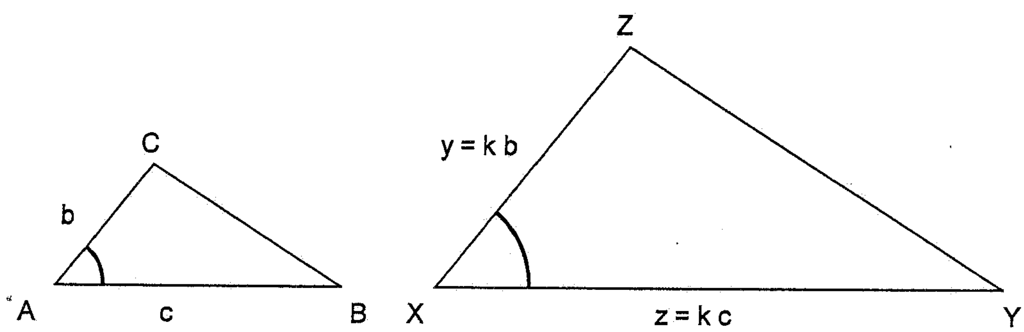
Theorem 2.2 **(Similarity by A.A.A.)**
If the three angles of one triangle are equal respectively to the three angles of another, then the two triangles are similar.

We use the letters A.A.A. to denote similarity established in this way.



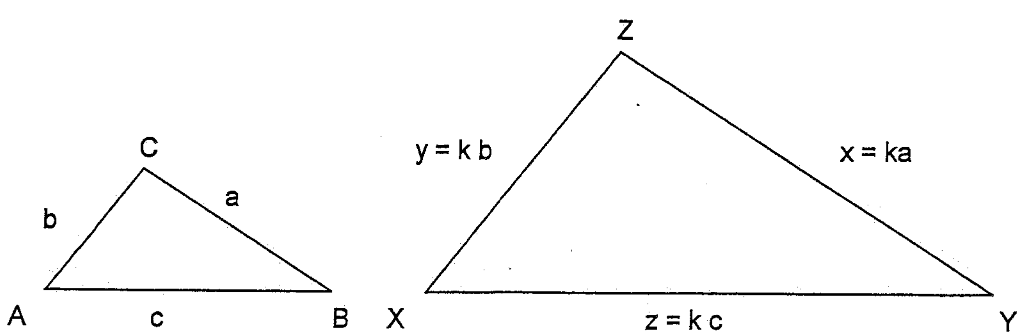
Theorem 2.3 **(Similarity by S.A.S.)**
If the ratio of one side of a triangle to one side of a second triangle is equal to the ratio of another pair of sides of the two triangles and the included angles of these sides in the triangles are equal, then the two triangles are similar.

We use the letters S.A.S. to denote similarity established in this way.

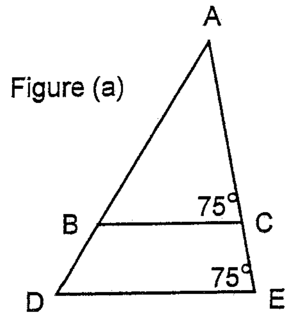


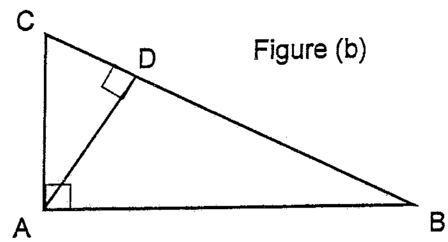
Theorem 2.4 **(Similarity by S.S.S.)**
If one can set up correspondence between the vertices of two triangles such that the ratio of corresponding sides is the same for all three sides, then the two triangles are similar.

We use the letters S.S.S. to denote congruence established in this fashion. Note that this theorem is the converse of theorem 2.1.



Example 2.5 How many similar triangles are there in the figure (a) below?

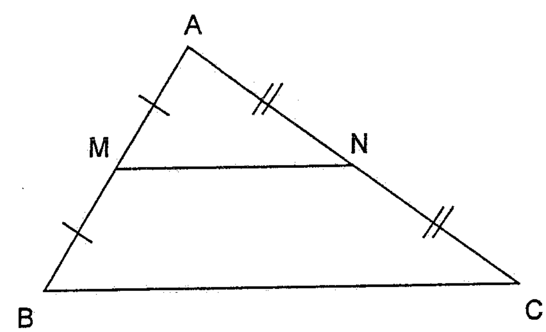




Example 2.6 How many similar triangles are there in the figure (b) above?

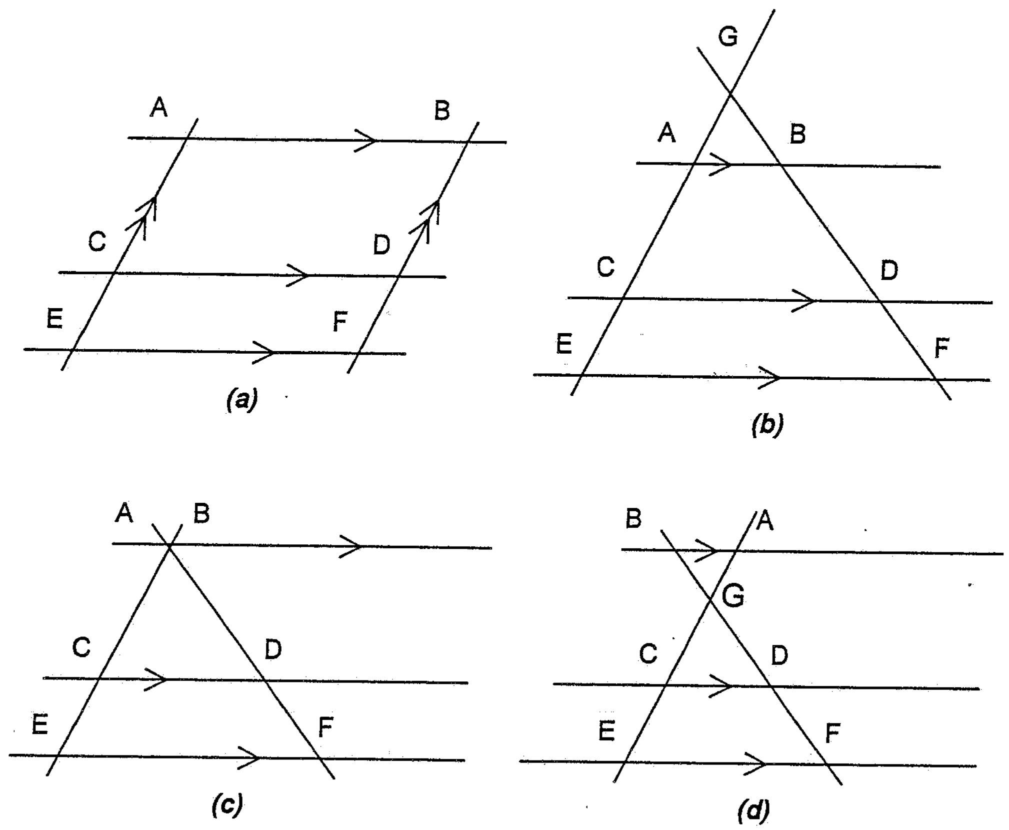
Using similar triangles, one can prove the midpoint theorem and intercept theorem, which are very useful in many different applications.

Theorem 2.7 **(Midpoint Theorem)**
The line segment joining the midpoints of two sides of a triangle is parallel to the remaining side with length half of that of this remaining side.



$AM = MB$ and $AN = NC \Rightarrow MN \,//\, BC$ and $MN = \frac{1}{2} BC$

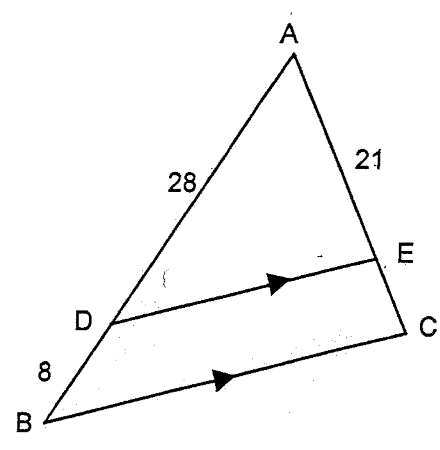
Theorem 2.8 **(Intercept Theorem)**
The lengths of line segments cut by three parallel lines on transversals are proportional.



$$\frac{AC}{CE} = \frac{BD}{DF}$$

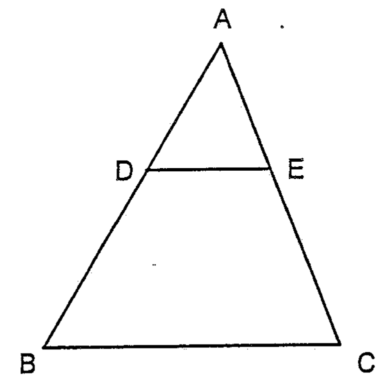
Note: The midpoint theorem can be proved as a corollary of the intercept theorem.

Example 2.9 Find EC.

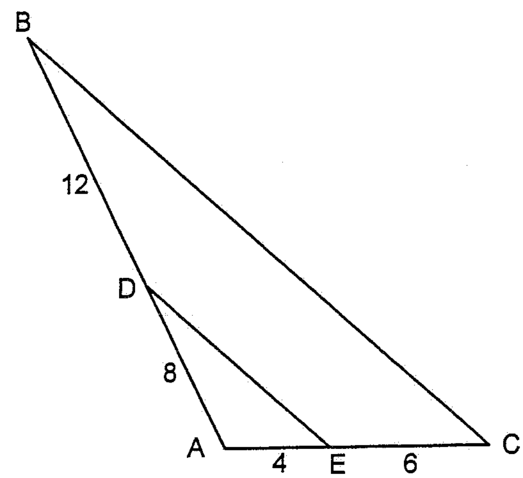


Theorem 2.10 **(Converse of the intercept theorem)**
If a line divides two sides of a triangle in the same ratio, then the line is parallel to the third side of the triangle.

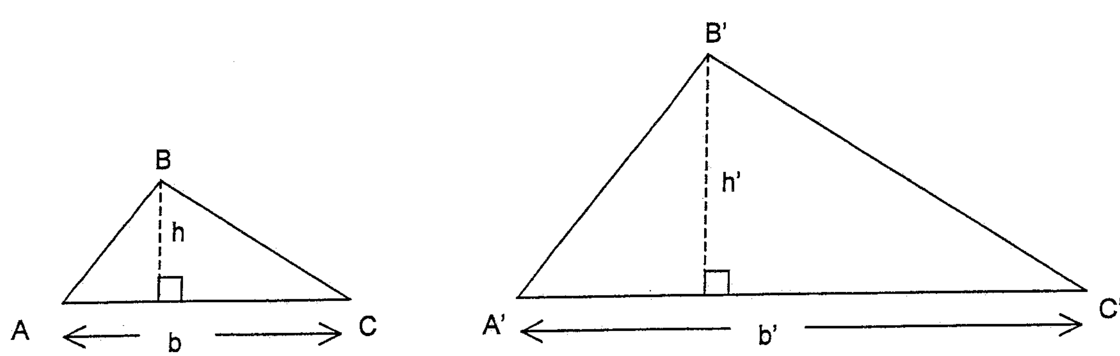
$$\frac{AD}{DB} = \frac{AE}{EC} \Rightarrow DE \,//\, BC$$



Example 2.11 Prove that DE // BC.



Area of Similar Figures
In theorem 2.1, we learned that the ratios of corresponding parts of similar triangles are equal. What can we say about the areas of two similar triangles? Consider the following two similar triangles.



By theorem 2.1, we have

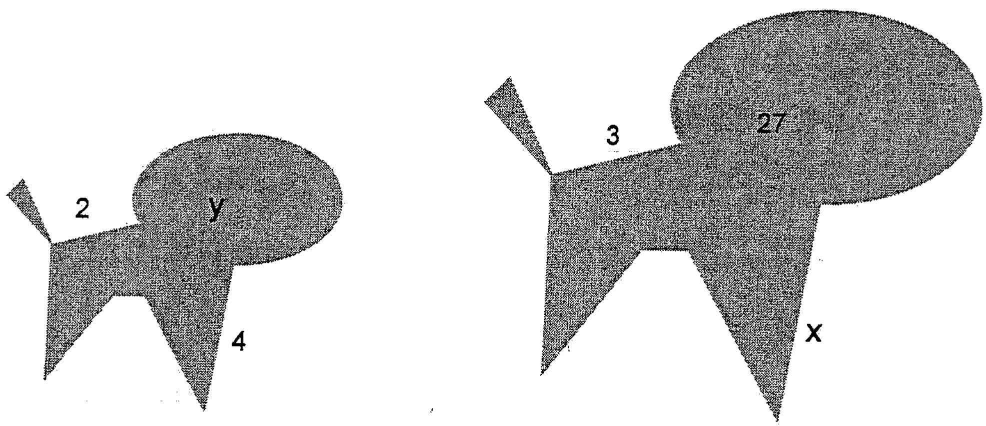
$$\frac{h'}{h} = \frac{b'}{b}$$

Applying theorem 1.7 (for area of triangle), the ratio of the areas of the two triangles is

$$\frac{\text{Area}(\Delta A'B'C')}{\text{Area}(\Delta ABC)} = \frac{\frac{1}{2}b'h'}{\frac{1}{2}bh} = \frac{b'h'}{bh} = \left(\frac{b'}{b}\right)\times\left(\frac{h'}{h}\right) = \left(\frac{b'}{b}\right)^2 \text{ or } \left(\frac{h'}{h}\right)^2.$$

Theorem 2.12 **The ratio of the areas of two similar triangles is equal to the square of the ratio of the lengths of a pair of corresponding parts.**

Note: We can generalize theorems 2.1 and 2.12 to similar figures of arbitrary shape (not necessarily triangles). For example, suppose the two objects shown below are similar.



By generalizing theorem 2.1, we have $\frac{x}{4} = \frac{3}{2} \Rightarrow 2x = 3(4) \Rightarrow x = 6$ (units)

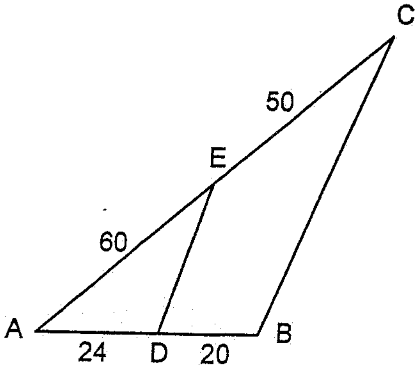
Similarly, generalization of theorem 2.12 yields the area y of the figure on the left:

$$\frac{y}{27} = \left(\frac{2}{3}\right)^2 \Rightarrow \frac{y}{27} = \frac{4}{9} \Rightarrow 9y = 4(27) \Rightarrow y = 12 \text{ (square units).}$$

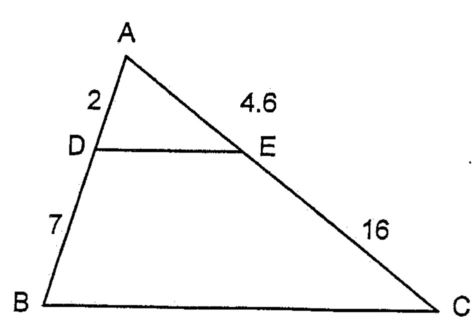
Exercise 2

In Questions 1 – 4, is DE // BC?

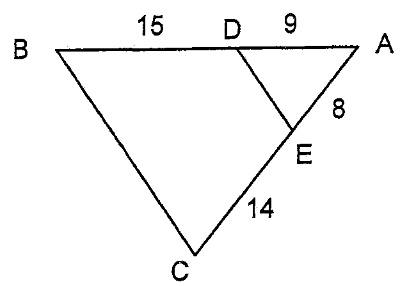
1.



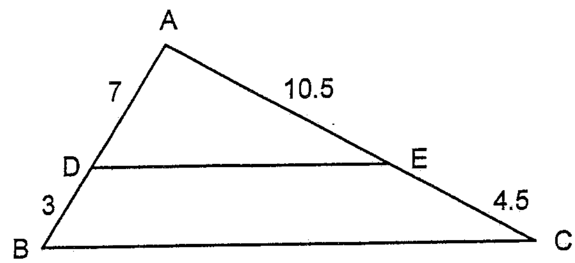
2.



3.



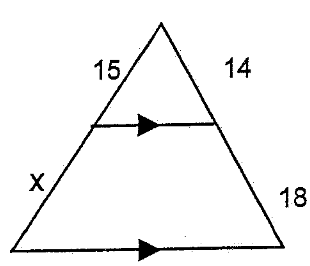
4.



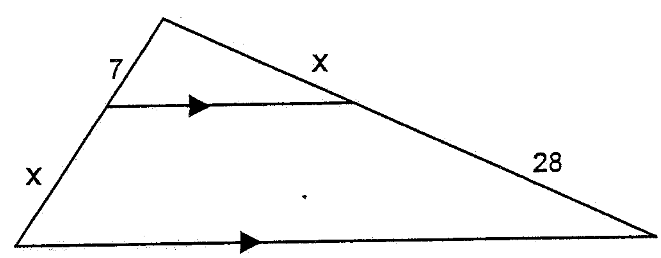
5. In the figure for question 1, if DE = 32, find the length of BC.
6. In the figure for question 2, all lengths unchanged but EC. What is the length of EC to make DE // BC?
7. In the figure for question 2, all lengths unchanged but AE. What is the length of AE to make DE // BC?
8. In the figure for question 3, all lengths unchanged but EC. What is the length of EC to make DE // BC?
9. In the figure for question 3, all lengths unchanged but BD. What is the length of BD to make DE // BC?
10. In the figure for question 4, if DE = 12, find the length of BC.

Find the unknown(s) in Questions 11 - 14.

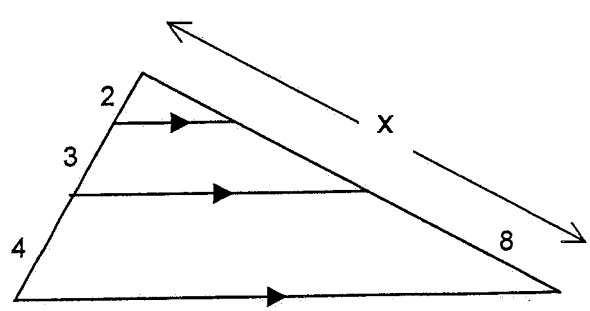
11.



12.



13.



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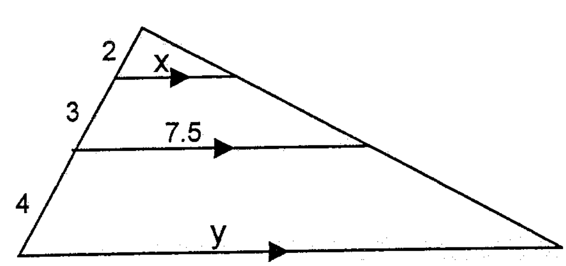
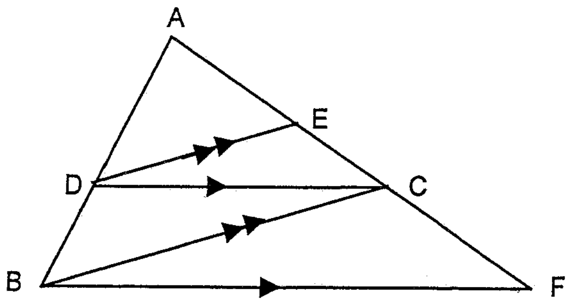
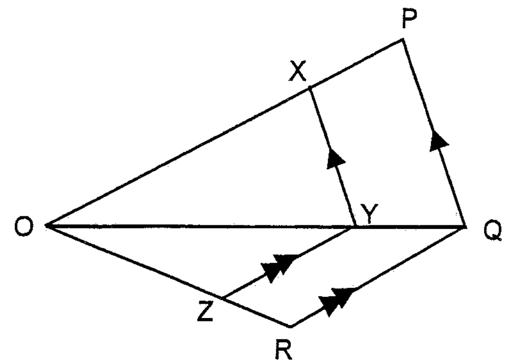


15. ADB and AECF are straight lines such that DE // BC, DC // BF.

Prove that $\dfrac{AE}{EC} = \dfrac{AC}{CF}$.

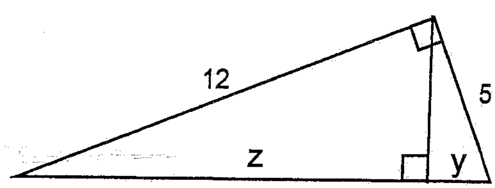




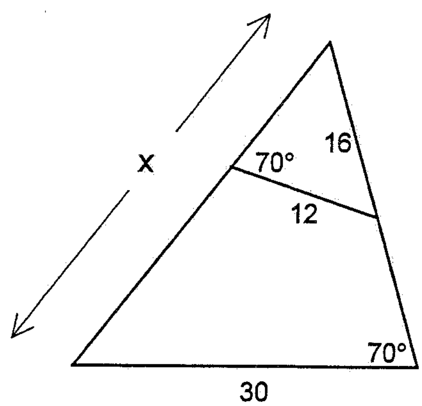
16. OXP, OYQ and OZR are straight lines such that XY //PQ and YZ // QR. Prove that XZ // PR.

Find the marked length(s) in Questions 17 - 21.

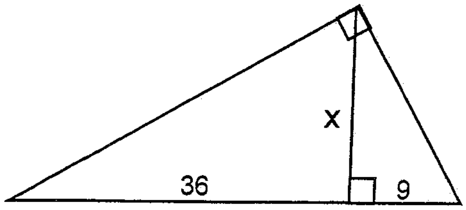
17.



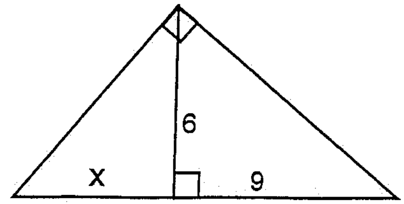
18.



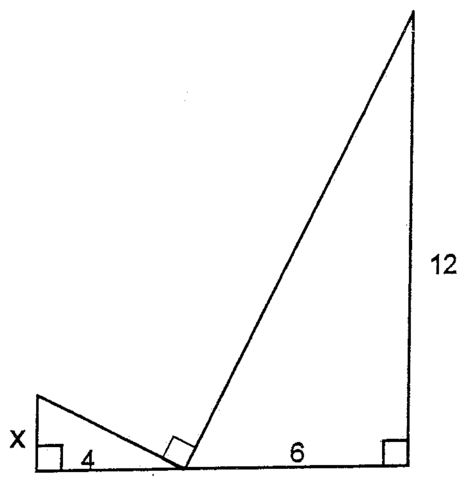
19.

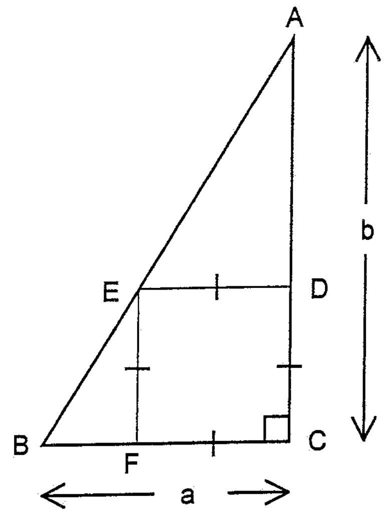


20.



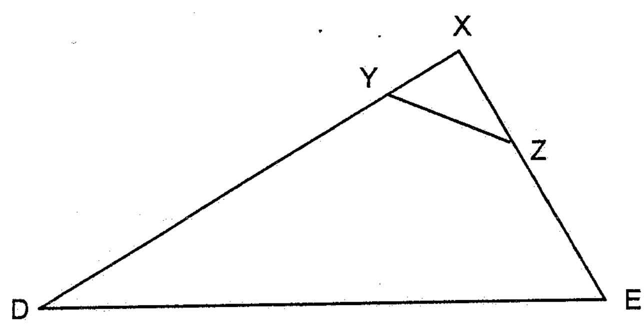
21.





22. ΔABC is a right triangle with the right angle at vertex C and CDEF is a square. Prove that $CD = \dfrac{ab}{a+b}$

23. XY = 2, XZ = 3, YZ = 4, ZE = 5 and YD = 10. Find the perimeter of ΔXDE.



24. Suppose a man 2 m tall casts a shadow 3 m long. At the same time a building casts a shadow 120 m long. What is the height of the building?

25. Suppose a stick, 3 m long, casts a shadow 4.5 m long. At the same time, a tree casts a shadow 20 m long. Find the height of the tree.

Each of the following is the ratio of lengths of corresponding sides of two similar triangles. In each case, evaluate the ratio of the areas of the triangles.

26. 5 : 3 27. 4 : 6 28. 3 : 7
29. 5 : 100 30. 3.5 : 6 31. 1 : 7
32. 4 : 1 33. 4 : 6 34. $1 : \sqrt{2}$

Each of the following is the ratio of lengths of corresponding sides of two similar polygons. In each case, evaluate the ratio of the areas of the polygons.

35. 9 : 1 36. 2 : 5 37. 3.5 : 7.5
38. 500 : 100 39. 3.5 : 700 40. 1000 : 7000
41. $5 : \sqrt{5}$ 42. $\sqrt{6} : 6$ 43. $1 : \sqrt{2}$

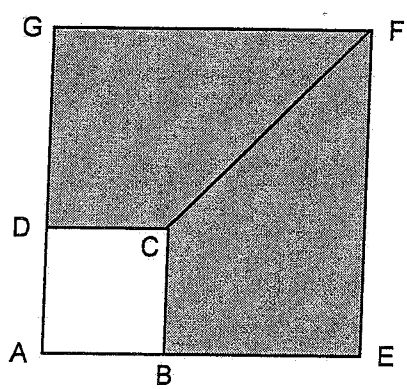
Each of the following is the ratio of the areas of two similar triangles. In each case, evaluate the ratio of the lengths of the corresponding side of the triangles.

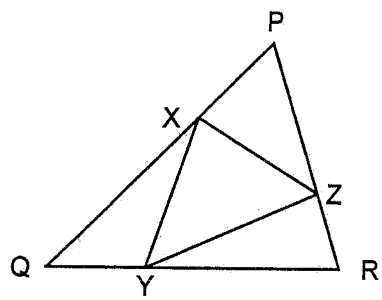
44.	9: 1	45.	16 : 25	46.	25 : 36
47.	1600 : 900	48.	144 : 25	49.	1 : 4
50.	100 : 100	51.	$\sqrt{52} : \sqrt{13}$	52.	$\sqrt{208} : \sqrt{13}$

53. If ΔABC ~ ΔXYZ, BC = 7 cm, YZ = 10 cm and the area of ΔXYZ is 42 cm^2, find the area of ΔABC

54. If the area of ΔABC is 25 cm^2, area of ΔDEF is 4 cm^2, AC = 3 cm and ΔABC ~ ΔDEF, find the length of DF.

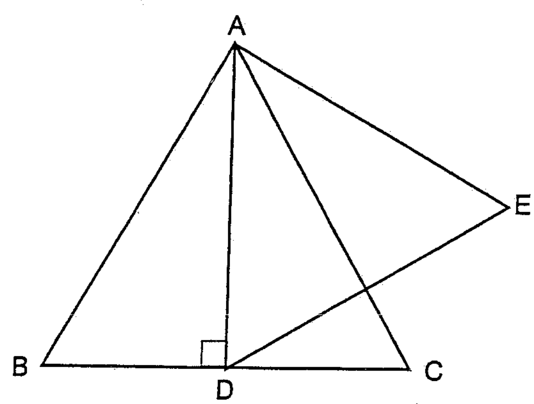
55. ABCD and AEFG are squares. If the areas of ABCD and the shaded part BEFGDC are respectively 25 cm^2 and 144^2 cm, find CF.





56*. $\frac{PX}{XQ} = \frac{QY}{YR} = \frac{RZ}{ZP} = \frac{1}{2}$. If the area of ΔPQR is 9 cm^2, find the area of ΔXYZ.

57. ΔABC and ΔADE are equilateral triangles. If the area of ΔABC is 3 cm^2, find the area of ΔADE.



58. Use the intercept theorem to prove the midpoint theorem.