Name: Solution Set

Signature: ______________________

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Mathematics Department

MATH 185
PRACTICE FINAL

You have three (3) hours to write this exam.

You may use your pen, pencil, eraser, ruler, protractor, and calculator. Use the last page of this examination for rough work. This page may be detached; however it must be included with your examination when you turn it in. You may not use your own scrap paper.

Your method of solution must be clearly shown and valid to obtain full marks for any problem.

GOOD LUCK!

1. Which of the following are vectors or scalars? Write "V" for vector, "S" for scalar, and "neither" if the quantity isn't a vector or a scalar. (3 points)

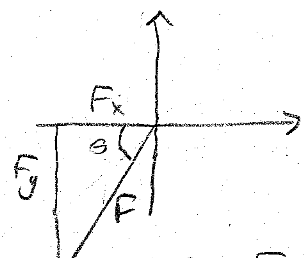
   a) -3 at $25°$ ____V____

   b) $15°$ west of north ____neither____

   c) 2.5 kg ____S____

2. State the magnitude and direction of the following vector. Round your answers to one decimal place. Show your work. (3 points)

   $F_x = -53$ N, $F_y = -125$ N



$$F = \sqrt{F_x^{\,2} + F_y^{\,2}}$$
$$= \sqrt{(53)^2 + (-125)^2}$$
$$= 135.772 \text{ N}$$
$$= 135.8 \text{ N}$$

$$\tan\theta = \frac{F_y}{F_x} = \frac{-125}{-53}$$

or coterminal $(-113.0°)$

$$\theta = \cancel{67.0°},\ 247.0°$$

wrong quadrant

$$\boxed{\vec{F} = 135.8\text{N at } 247.0°}$$

3. Calculate the magnitude of the vector **A** = 5**i** – 4**j** – 2**k** using the dot product. Start with the definition of dot product, and give an exact answer. (2 points)

$$\vec{A}\cdot\vec{A} = A_x A_x + A_y A_y + A_z A_z$$
$$A^2 = 5^2 + (-4)^2 + (-2)^2$$
$$= 45$$
$$\boxed{A = 3\sqrt{5}}$$

4. Calculate the cross product **A**×**B** for the following vectors. Show your work.(2 points)

A = –2**i** + 3**k**, **B** = **j** + **k**

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ -2 & 0 \\ 0 & 1 \end{matrix} = -2\hat{k} - (3\hat{i} - 2\hat{j})$$

$$= -3\hat{i} + 2\hat{j} - 2\hat{k}$$

$$\boxed{\vec{A} \times \vec{B} = -3\hat{i} + 2\hat{j} - 2\hat{k}}$$

5. State the values of x for which the function below is continuous. (2 points)

$$f(x) = \frac{1}{x^2 - 5x} = \frac{1}{x(x-5)}$$

$x \neq 0$ and $x \neq 5$

$(-\infty, 0) \cup (0, 5) \cup (5, \infty)$

6. Evaluate the following limit. Show your work. (2 points)

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{x\cancel{(x-1)}}$$

$$= \lim_{x \to 1} \frac{(x+1)}{x}$$

$$= \frac{2}{1} = 2$$

2

7. Find the derivative $\frac{dy}{dx}$ for the following expressions. Tidy up your final answer, but you'll find that these won't simplify much. Show your work. (16 points)

a) $y = \ln\sqrt{4\cos x - 3}$ $= \frac{1}{2}\ln(4\cos x - 3)$ $\frac{dy}{dx} = \frac{-2\sin x}{4\cos x - 3}$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{4\cos x - 3}\right)4(-\sin x)$$

$$= \frac{-2\sin x}{4\cos x - 3}$$

b) $y = e^{3/x^2}\sin(2x^2 - 1)$ see below

$$\frac{dy}{dx} = e^{3/x^2}\cos(2x^2-1)\,4x + \sin(2x^2-1)\,e^{3/x^2}(-2)3x^{-3}$$

$$= 4xe^{3/x^2}\cos(2x^2-1) - \frac{6e^{3/x^2}\sin(2x^2-1)}{x^3}$$

c) $y = \sec x - \ln^2(x^2 - 1)$ $\frac{dy}{dx} = \sec x\tan x - \frac{4x\ln(x^2-1)}{x^2-1}$

$$\frac{dy}{dx} = \sec x\tan x - 2\ln(x^2-1)\left(\frac{1}{x^2-1}\right)2x$$

$$= \sec x\tan x - \frac{4x\ln(x^2-1)}{(x^2-1)}$$

d) $3\ln(xy) + \sin y = x^2$ $\frac{dy}{dx} = \frac{2x^2y - 3y}{3x + xy\cos y}$

$$\frac{3}{xy}\frac{d}{dx}(xy) + \cos y\frac{dy}{dx} = 2x$$

$$\frac{3}{xy}\left(x\frac{dy}{dx} + y\right) + \cos y\frac{dy}{dx} = 2x$$

$$xy\left(\frac{3}{y}\frac{dy}{dx} + \frac{3}{x} + \cos y\frac{dy}{dx}\right) = (2x)xy$$

$$3x\frac{dy}{dx} + 3y + xy\cos y\frac{dy}{dx} = 2x^2y$$

$$(3x + xy\cos y)\frac{dy}{dx} = 2x^2y - 3y$$

$$\boxed{\frac{dy}{dx} = \frac{2x^2y - 3y}{3x + xy\cos y}}$$

8. Calculate the following derivative with respect to x. To receive full marks, write as a single fraction and simplify fully, showing your work. (4 points)

$$y = \frac{\sqrt[3]{x}}{5-x^2}$$

$$\frac{dy}{dx} = \frac{(5-x^2)\frac{1}{3}x^{-2/3} - x^{1/3}(-2x)}{(5-x^2)^2}$$

multiply by $\frac{3x^{2/3}}{3x^{2/3}}$

$$= \left(\frac{1}{5-x^2}\right)^2 \left(\frac{5-x^2}{3x^{2/3}} + 2x^{4/3}\right)$$

$$= \frac{1}{(5-x^2)^2}\left(\frac{5-x^2+6x^2}{3x^{2/3}}\right) = \frac{5x^2+5}{3\sqrt[3]{x^2}(5-x^2)^2} \text{ or } \frac{5(x^2+1)}{3\sqrt[3]{x^2}(5-x^2)^2}$$

9. Using the **definition of derivative**, find the derivative of the function below with respect to x. Show your work. (6 points)

a) $y = \dfrac{1}{x+2}$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h+2}\frac{(x+2)}{(x+2)} - \frac{1}{x+2}\frac{(x+h+2)}{(x+h+2)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{x+2-(x+h+2)}{(x+h+2)(x+2)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{x+2-x-h-2}{(x+h+2)(x+2)}\right) = \lim_{h\to 0} \frac{1}{\cancel{h}}\left(\frac{-\cancel{h}}{(x+h+2)(x+2)}\right)$$

$$= \lim_{h\to 0} \frac{-1}{(x+h+2)}\frac{1}{(x+2)}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{(x+2)^2}}$$

b) For what values of x is the function $f(x)$ in part (a) differentiable?

$x \neq -2$

10. Consider the curve given by the function $f(x) = \tan^{-1}(x)$. For what values of x (if any) does the curve have a slope of ½? (4 points)

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{2}$$

$$\frac{1}{1+x^2} = \frac{1}{2}$$

$$2 = 1 + x^2$$

$$1 = x^2$$

$$\boxed{x = \pm 1}$$

11. Consider the function $f(x) = e^x + x$. Use Newton's method to calculate the root of the function to three decimal places. (Hint: calculate *f(x)* for $x = -1$, 0, and 1 to determine a reasonable starting point.) (4 points)

$$f(0) = e^0 + 0 = 1$$

$$f(1) = e^1 + 1 \approx 3.718...$$

$$f(-1) = e^{-1} - 1 \approx -0.632$$

so the root is between -1 and 0

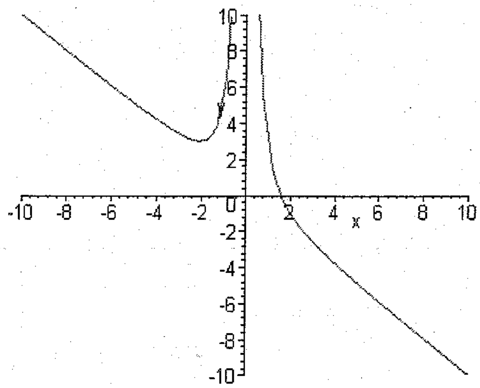
$$f'(x) = e^x + 1$$

x	$f(x)$	$f'(x)$	$x - \frac{f(x)}{f'(x)}$
-0.5	0.106531	1.60653	-0.566311
-0.566311	0.001305	1.56762	-0.567143
-0.567143	4.55114×10^{-7}	1.56714	-0.567143

$\left.\begin{matrix} -0.567143 \\ -0.567143 \end{matrix}\right\}$ same

$$\boxed{x = -0.567}$$

12. The function $y = \frac{4}{x^2} - x$ is shown in the curve below. (12 points)



a) Find the y-intercept and zeros of this function, if any.

y-int: set $x = 0$ $y = \frac{4}{0^2} - x$ $=$ undefined

y-intercept: none

zeros: $x = \sqrt[3]{4}$

x-int: set $y = 0$ $0 = \frac{4}{x^2} - x$

$x = \frac{4}{x^2}$

$x^3 = 4$

$x = \sqrt[3]{4} \approx 1.5874$

b) Find the asymptotes of this function, if any.

vertical $\boxed{x = 0}$

oblique: as $x \to \infty$ $\frac{4}{x^2} \to 0$, $-x \to \infty$ } first term disappears so asymptote at $y = -x$

c) Find the critical points of this function, if any. If there are critical points, use one of the derivative tests to verify whether these points are relative maxima or minima, and show your work.

$y = 4x^{-2} - x$

$0 = y' = 4(-2)x^{-3} - 1$

$0 = \frac{-8}{x^3} - 1$

$1 = -\frac{8}{x^3}$

$x^3 = -8$

$x = -2$

at $x = -2$

$y = \frac{4}{4} - (-2) = 3$

$\boxed{(-2, 3) \text{ is a critical point}}$

$y' = -8x^{-3} - 1$

$y'' = 24x^{-2}$

$= \frac{24}{x^2} = +$ when $x = -2$

$\therefore$ concave up

$\boxed{\text{minimum}}$

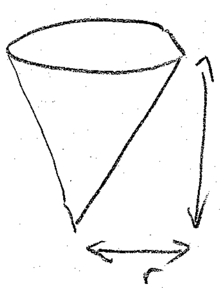
d) Find the inflection points of this function, if any.

$y'' = \frac{24}{x^2}$ ← never equals zero

$\therefore$ no inflection points

13. A cone-shaped tank has a height that is equal to its radius. If the tank is being filled with water at a rate of 3 m^3 per minute, calculate how fast the water level is rising when the tank contains 120 m^3 of water. (The volume of a cone equals one-third of the volume of the cylinder with the same height and radius.) (6 points) → $\frac{dh}{dt}$

note: I should have said it was an inverted cone - sorry!



$h = r$

$V_{cone} = \frac{1}{3}\pi r^2 h$

we want $\frac{dh}{dt}$, so substitute h for r:

$V_{cone} = \frac{1}{3}\pi h^3$

$\frac{dV}{dt} = \frac{1}{3}\pi(3h^2)\frac{dh}{dt}$

$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{\pi h^2}\frac{dV}{dt} = \frac{1}{\pi(4.857)^2}(3) = 0.04\ m/min$

$\frac{dV}{dt} = 3\ \frac{m^3}{min}$

$V = \frac{1}{3}\pi h^3$

$120 = \frac{1}{3}\pi h^3$

$\frac{360}{\pi} = h^3$

$h = 4.857$ m

14. Use differentials or a linear approximation to estimate the value of $\sqrt{16.04}$. Show all of your steps. (5 points)

$y = \sqrt{x}$ where $x = 16$, $dx = 0.04$

$dy = \frac{dy}{dx}\,dx$

$= \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$

$= \frac{1}{2}\frac{1}{\sqrt{16}}(0.04)$

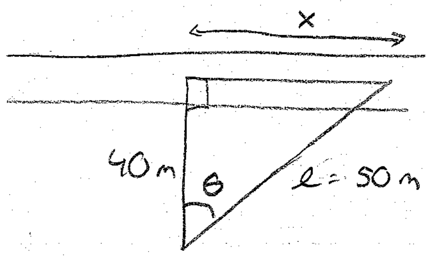
$= 0.005$

so $\sqrt{16.04} = \sqrt{16} + dy$

$= 4 + 0.005$

$= 4.005$

15. I am standing 40 m from a straight road watching a cyclist on the road through my binoculars. At the instant the cyclist is 50 m away from me, my binoculars are rotating at 0.3 radians per second. How fast is the cyclist moving? (5 points)



when $\ell = 50\,m$,

$x = 30\,m$

$\cos\theta = \frac{40}{50}$

$$\tan\theta = \frac{x}{40}$$

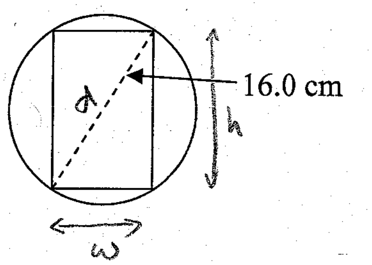
$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{40}\frac{dx}{dt}$$

$$v_x = \frac{dx}{dt} = 40\sec^2\theta \frac{d\theta}{dt}$$

$$= \frac{40}{\cos^2\theta}\frac{d\theta}{dt}$$

$$v_x = \frac{40}{(40/50)^2}(0.3) = 18.75\ m/s$$

16. The strength S of a beam with a rectangular cross section is given by $S = kwh^2$, where w is the width of the beam, h is the height of the beam, and k is a constant. Find the dimensions of the strongest beam that can be cut from a cylindrical log which is 16.0 cm in diameter. (8 points)



$$d^2 = w^2 + h^2$$

$$16^2 = w^2 + h^2$$

$$h^2 = 256 - w^2$$

$$S = kwh^2$$

$$S = kw(256 - w^2) = 256kw - kw^3$$

$$0 = \frac{dS}{dw} = 256k - 3kw^2$$

$$3kw^2 = 256k$$

$$3w^2 = 256$$

$$w^2 = \frac{256}{3}$$

$$w = \pm 9.2376 = 9.2\ cm$$

discard neg answer

$$h = \sqrt{256 - w^2} = 13.1\ cm$$

check that it's actually a max:

$$\frac{d^2S}{dw^2} = -6kw$$

for $+w$, always $-$

$\therefore$ concave down

$\frown$

max ✓

$\boxed{\text{The width is 9.2 cm and the height is 13.1 cm.}}$

17. Find, if possible, A+B, AB, and BA. If the result is undefined, say so. (6 points)

a) $\underset{1\times 3}{A=\begin{bmatrix}8 & 0 & -2\end{bmatrix}}, \underset{1\times 1}{B=\begin{bmatrix}-1\end{bmatrix}}$

A + B = not possible

AB = not possible

$BA = \begin{bmatrix}-1\end{bmatrix}\begin{bmatrix}8 & 0 & -2\end{bmatrix} = \begin{bmatrix}-8 & 0 & +2\end{bmatrix}$

b) $\underset{1\times 2}{A=\begin{bmatrix}1 & -2\end{bmatrix}}, \underset{2\times 1}{B=\begin{bmatrix}3\\-4\end{bmatrix}}$

A + B = not possible

$AB = \begin{bmatrix}3+8\end{bmatrix} = \begin{bmatrix}11\end{bmatrix}$

$BA = \begin{bmatrix}3 & -6\\-4 & 8\end{bmatrix}$

18. Calculate the determinant of the following matrices. Then find the inverse matrix, if it exists. You don't need to show any work for this one. (4 points)

a) $\begin{bmatrix}20 & -2\\-5 & -1\end{bmatrix} = A$

det A = -30

$A^{-1} = -\frac{1}{30}\begin{bmatrix}-1 & 2\\5 & 20\end{bmatrix} = \begin{bmatrix}1/30 & -\frac{1}{15}\\-1/6 & -2/3\end{bmatrix}$

b) $\begin{bmatrix}1 & -2 & 0\\-2 & 4 & 8\\3 & -6 & 6\end{bmatrix} = B$

det B = 0

B^{-1} does not exist

19. Solve the following systems of equations using the specified method. Show your work! (6 points)

a) $\begin{aligned} -2x+3y&=6 \\ 3x-5y&=-11 \end{aligned}$ using Cramer's rule $\underline{(3,4)}$

$D = \begin{vmatrix} -2 & 3 \\ 3 & -5 \end{vmatrix} = 1$

$x = \dfrac{D_x}{D} = 3$

$D_x = \begin{vmatrix} 6 & 3 \\ -11 & -5 \end{vmatrix} = 3$

$y = \dfrac{D_y}{D} = 4$

$D_y = \begin{vmatrix} -2 & 6 \\ 3 & -11 \end{vmatrix} = 4$

b) $\begin{aligned} 4x-12y+7z&=4 \\ -2x+7y-4z&=0 \\ x-3y+2z&=-1 \end{aligned}$ by using an inverse matrix $\underline{(9,-2,-8)}$

$AX = B$ where $A = \begin{bmatrix} 4 & -12 & 7 \\ -2 & 7 & -4 \\ 1 & -3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$

$X = A^{-1}B$ where $A^{-1} = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 9 \\ -2 \\ -8 \end{bmatrix}$