Math 185 – Quiz #4

December 3, 2007 **Name:** Solution Set

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Total: 30 points

1. Find the derivative $\frac{dy}{dx}$ for the following functions.

a) $y = \frac{\sec x}{x}$ (3 points)

$$\frac{dy}{dx} = \frac{x \sec x \tan x - \sec x(1)}{x^2}$$

$$= \frac{\sec x(x \tan x - 1)}{x^2} \quad \text{or} \quad \left(\frac{\sec x \tan x}{x} - \frac{\sec x}{x^2}\right)$$

b) $y = \ln\left(x \tan^{-1} x\right) = \ln x + \ln(\tan^{-1} x)$ (4 points)

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\tan^{-1} x}\left(\frac{1}{1+x^2}\right)$$

(1) derivative of ln
(2) product rule / properties of logs
(1) derivative of $\tan^{-1} x$

or $y = \ln(x \tan^{-1} x)$

$$\frac{dy}{dx} = \frac{1}{x \tan^{-1} x}\left(x \cdot \frac{1}{1+x^2} + \tan^{-1} x \cdot (1)\right)$$

$$= \frac{1}{x \tan^{-1} x}\left(\frac{x}{1+x^2} + \tan^{-1} x\right) \Leftarrow$$ which if you multiply out gives you the previous answer

$$= \frac{x + (1+x^2)\tan^{-1} x}{x(1+x^2)\tan^{-1} x}$$ (if you insist on a common denominator)

c) $e^y = y\sin 2x$ (5 points)

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(y \sin 2x)$$

$$e^y \frac{dy}{dx} = y(\cos 2x)(2) + \sin 2x \frac{dy}{dx}$$

$$e^y \frac{dy}{dx} - \sin 2x \frac{dy}{dx} = 2y \cos 2x$$

$$\frac{dy}{dx} = \frac{2y \cos 2x}{e^y - \sin 2x}$$

or

$$y = \ln(y \sin 2x)$$

$$y = \ln y + \ln(\sin 2x)$$

$$\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx} + \frac{1}{\sin 2x}(\cos 2x)\cdot 2$$

$$\frac{dy}{dx} - \frac{1}{y}\frac{dy}{dx} = \frac{2\cos 2x}{\sin 2x}$$

$$y\frac{dy}{dx} - \frac{dy}{dx} = \frac{2y\cos 2x}{\sin 2x}$$

$$(y-1)\frac{dy}{dx} = 2y \cot 2x$$

$$\frac{dy}{dx} = \frac{2y\cot 2x}{y-1}$$

2. Find the derivative $f'(x)$ for the following function. It should simplify nicely.(5 points)

$f(x) = 2x\sin x + 2\cos x - x^2\cos x$

$$f'(x) = 2x\cos x + 2\sin x(1) + 2(-\sin x) - x^2(-\sin x) - \cos x(2x)$$

$$= 2x\cos x + 2\sin x - 2\sin x + x^2\sin x - 2x\cos x$$

$$= x^2 \sin x$$

3. Rewrite $y = \cot x$ in terms of other trig functions and then differentiate in order to **derive** the rule for differentiating the cotangent function. (3 points)

$$y = \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\csc^2 x \quad (\text{since } \sin^2 x + \cos^2 x = 1)$$

$$\text{so } \frac{d}{dx}(\cot \Box) = -\csc^2 \Box \frac{d\Box}{dx}$$

or

$$y = \frac{1}{\tan x} = (\tan x)^{-1}$$

$$\frac{dy}{dx} = -(\tan x)^{-2} \sec^2 x$$

$$= -\frac{1}{\tan^2 x} \sec^2 x$$

$$= -\frac{\cancel{\cos^2 x}}{\sin^2 x} \cdot \frac{1}{\cancel{\cos^2 x}}$$

$$= -\frac{1}{\sin^2 x}$$

$$= -\csc^2 x \quad (\text{again!})$$

(-1) not simplified

4. If $\ln 2 = 0.693$ (to three decimal places), estimate $\ln 2.002$. (Hint: use a differential to estimate the change in $y = \ln x$ as x increases from 2 to 2.002.) (4 points)

$$y = \ln x$$

$$\frac{dy}{dx} = \frac{1}{x} \quad (1)$$

$$x = 2$$

$$\Delta x = 0.002 \quad (1)$$

$$dy = \frac{dy}{dx}\, dx$$

$$= \frac{1}{x}\, dx$$

$$= \frac{1}{2}(0.002)$$

$$= 0.001 \quad (1)$$

if $y = \ln x$ increases by 0.001 as x goes from 2 to 2.002

$$\text{then } \ln 2.002 = \ln 2 + 0.001$$

$$= 0.693 + 0.001$$

$$= 0.694 \quad (1)$$

5. A passenger on a ferris wheel has coordinates given by $x = 10\cos\theta$ and $y = 10\sin\theta$, where the coordinates x and y are given in metres. If the ferris wheel is rotating such that the angle is changing at a constant rate of $\frac{d\theta}{dt} = 0.2$ rads/s, find

a) the **magnitude** and **direction** of the velocity of the passenger when $\theta = \pi/2$
b) the **magnitude** of the velocity of the passenger for any angle θ.

(6 points)

a) $v_x = \frac{dx}{dt}$ so $x = 10\cos\theta$

$\frac{dx}{dt} = 10(-\sin\theta)\frac{d\theta}{dt}$

$v_x = \frac{dx}{dt} = -10\sin\theta\frac{d\theta}{dt}$

$= -10\sin\frac{\pi}{2}(0.2)$

$= -2$ m/s when $\theta = \pi/2$ 1

$v_y = \frac{dy}{dt}$ so $y = 10\sin\theta$

$\frac{dy}{dt} = 10\cos\theta\frac{d\theta}{dt}$

$v_y = \frac{dy}{dt} = 10\cos\theta\frac{d\theta}{dt}$

$= -10\cos\frac{\pi}{2}(0.2)$ 1

$= 0$ when $\theta = \pi/2$

at $\theta = \pi/2$, $v_x = -2$ m/s, $v_y = 0$ so $v = \sqrt{v_x^2 + v_y^2} = 2$ m/s 1

in the $-x$ direction 1

(or $180° = \pi$ if you prefer)

b) at any angle 2

$v_x = -10\sin\theta(0.2) = -2\sin\theta$

$v_y = 10\cos\theta(0.2) = 2\cos\theta$

$v = \sqrt{v_x^2 + v_y^2}$

$= \sqrt{(-2\sin\theta)^2 + (2\cos\theta)^2}$

$= \sqrt{4\sin^2\theta + 4\cos^2\theta}$

(unsimplified) -1

$= \sqrt{4}$ since $\sin^2\theta + \cos^2\theta = 1$

$= 2$ m/s

<u>speed</u> is constant because $\omega = \frac{d\theta}{dt} = $ constant and $v = r\omega$ for circular motion!