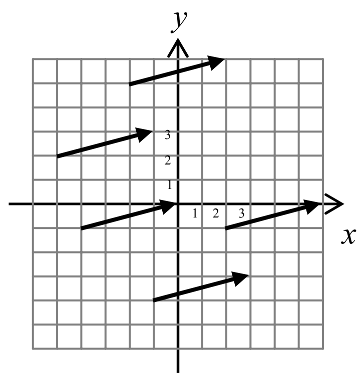
Section V.2: Magnitudes, Directions, and Components of Vectors

Vectors in the plane

If we graph a vector in the coordinate plane instead of just a grid, there are a few things to note. Firstly, directions will most likely be specified as angles in standard position (positive angles rotate counterclockwise, starting from the positive x-axis). Secondly, the vector does not need to begin or end at the origin. For example, all of the vectors in the diagram below are equal, because each arrow has the same magnitude and direction.



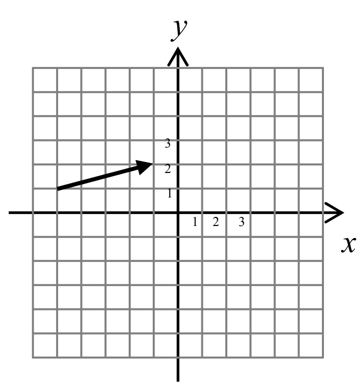
A vector may be moved all around the coordinate plane, and provided that the length and orientation of the arrow does not change, the vector itself will be considered unchanged.

Calculating magnitudes of vectors

To calculate the magnitude (length) of a vector, we may use some plane geometry. First, sketch your vector on the coordinate plane. Because the angle between the x- and y-axes is 90º, it should be straightforward to draw a right triangle with the arrow as hypotenuse and sides parallel to the axes. Then the length of the hypotenuse may be calculated from the lengths of the sides using the Pythagorean theorem, $a^2 + b^2 = c^2$, as in the example below.

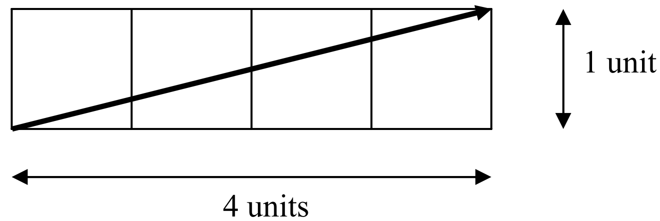
Example

What is the magnitude of vector **A** drawn in the diagram below?



Answer:

If you zoom in on the arrow, the blown-up picture looks like:



where the vector arrow is the hypotenuse of a right triangle that has a horizontal side with length of 4 units and vertical side of length 1 unit.

Plugging this into the Pythagorean theorem:

$$c^2 = a^2 + b^2$$
$$\mathrm{V}^2 = 4^2 + 1^2$$
$$\mathrm{V}^2 = 17$$
$$\mathrm{V} = \sqrt{17}$$

Since the magnitude of a vector is like a length measurement, the magnitude is always positive.

If you wish, you could also plug the coordinates into the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. As this formula is derived by putting generic coordinate points into the Pythagorean theorem, the calculation is essentially identical to the geometric method used in the example. In the example above, if you plug the points $(x_1, y_1) = (-5,1)$ and $(x_2, y_2) = (-1,2)$ into the distance formula, you will calculate the same result, $\mathrm{V} = \sqrt{17}$.

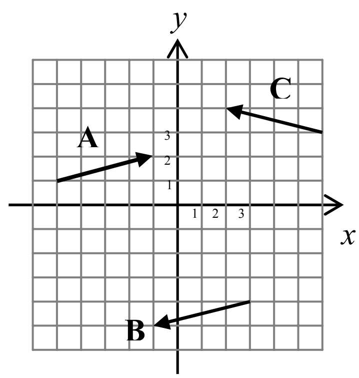
Calculating directions of vectors

When using a coordinate plane, directions should be specified using angles in standard position (positive angles rotate counterclockwise, starting from the positive *x*-axis). Remember that adding an integral multiple of 360º gives an angle that's coterminal to the first. Therefore, the direction given by the angle –20º (20º below the positive *x*-axis) could be equivalently specified by the angle 340º.

To determine the direction of a vector, graph it in the coordinate plane and consider once again the right triangle formed with the arrow as hypotenuse. Use right-triangle trig to determine the reference angle between the arrow and the *x*-axis, and then calculate the size of the angle in standard position.

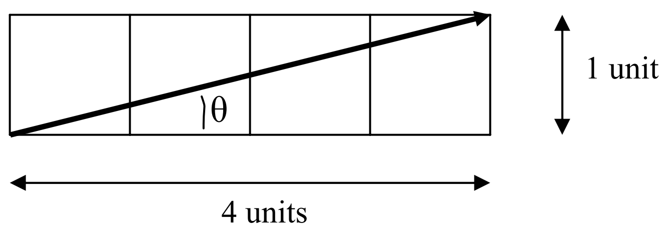
Example

Calculate the direction of the vectors **A, B,** and **C** in the diagram below.



Answer:

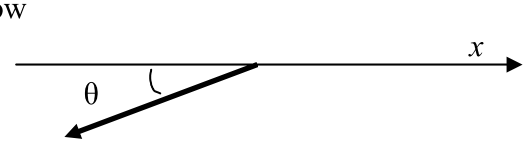
When you zoom in on vector **A**, you see the following.



The angle θ may be calculated from the arctangent function:

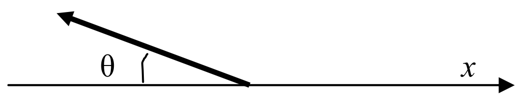
$$\tan\theta = \frac{y}{x}$$
$$\tan\theta = \frac{1}{4}$$
$$\theta = \tan^{-1}\left(\frac{1}{4}\right)$$
$$\theta = 14^\circ$$

You'll notice from the diagram that vector **B** will have a right triangle that's congruent to **A**'s triangle, and so will have the same reference angle (positive angle with respect to the *x*-axis). However, **B** is pointed in the opposite direction, so to get the angle in standard position, we consider the sketch below



and add 180º to **B**'s reference angle, to get the angle of 194º in standard position. If you prefer, you can consider the *x*- and *y*-coordinates of **B**'s triangle to be negative, given the arrow's orientation, but then you'll just have –1/–4 inside the tan function, which reduces to ¼, and you'll get the same result. Remember that the $\tan^{-1}(x)$ function only gives answers in the first and fourth quadrant, so if your vector has an angle in the second or third quadrant, you'll have to do a further calculation.

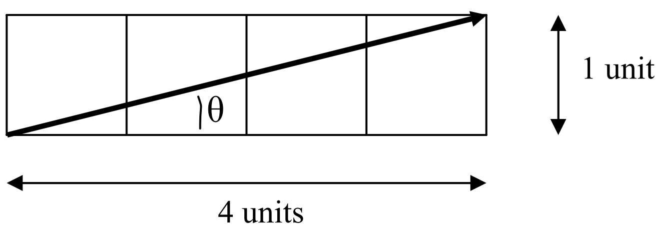
Vector **C** will also have a reference angle of 14º, and by consulting the sketch below, you can see that to get the angle in standard position, you'll have to subtract 14º from 180º to get a direction of 166º.



If, instead, you used –4 as your *x*-coordinate, you'll calculate an angle of –14º, which is in quadrant IV. By adding 180º to it to get an angle in QII, you'll arrive at the same 166º as found earlier, which is reassuring.

Component Form of a Vector

Looking closely at our vector **A** from previous examples, we have determined that its magnitude is $\sqrt{17}$ and its direction is 14º in standard position. However, this is a bit unsatisfactory in that the angle is an approximation only (though theoretically we could write it out with as many decimal places as we wanted). However, we do know the dimensions of the vector in the *x*- and *y*-directions <u>exactly</u>, as show in the diagram below.

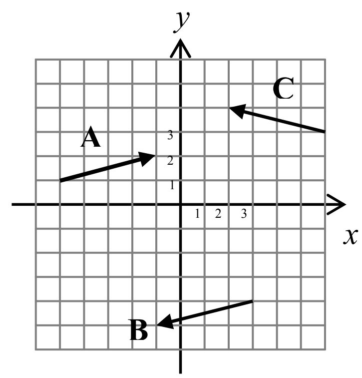


So rather than giving an approximation for this vector, it would be nice if there were a way to state that this vector was 4 units long in the +*x*-direction and 1 unit long in the +*y*-direction. This description is the <u>component form</u> of a vector.

There are a number of different notations for writing a vector in component form (also known as "resolving a vector into components"). For the purposes of this course, we will use the same notation as your physics and statics courses. Using this notation, the *x*-component of the vector **A** is called A_x and the *y*-component A_y. Our example above could then be described as the vector with $A_x = 4$ and $A_y = 1$. Note that although the magnitude of vector **A** is always positive, the individual components may be negative.

Example

Give the component form of vectors **B** and **C** in the diagram below.



Answer: By looking at the diagram, we can see that $B_x = -4$ and $B_y = -1$, while $C_x = -4$ and $C_y = 1$.

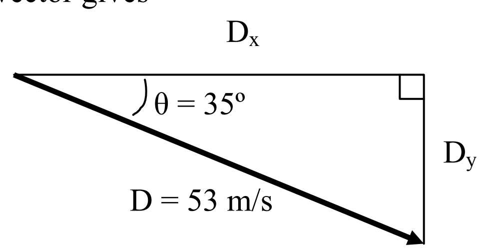
If a vector is given as a magnitude and direction, right-triangle trigonometry may be used to resolve the vector into components.

Example

Give the component form of the vector **D** = 53 m/s at –35º.

Answer:

A quick sketch of the vector gives



Note that an unbolded D is used for the hypotenuse, since we are referring only to the vector's magnitude.

From right-triangle trig,

$$\cos\theta = \frac{D_x}{D}$$
$$\begin{aligned} D_x &= D\cos\theta \\ &= 51\cos 35^\circ \\ &\cong 41.7768 \\ &\cong 42 \end{aligned}$$

and, similarly, $D_y = D \sin \theta = 51 \sin 35^\circ \cong 29.2524 \cong 29$. As D_y points downward, though, the y-component should be negative, so you would therefore write as your final answer that $D_x = 42$ m/s and $D_y = -29$ m/s, giving the components the same units as the original vector magnitude.

There is one other notation for component form that is commonly used in science and engineering textbooks. But in order to understand how it works, we must first look at the ideas of scalar multiplication and unit vectors.

Multiplication of a vector by a scalar

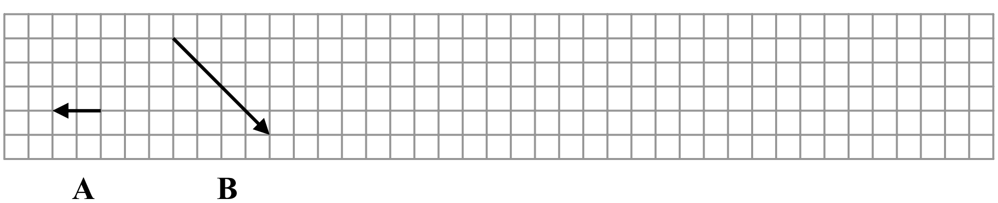
If a vector **A** is multiplied by a positive real number *b*, then the product *b***A** is a vector with in the same direction as the original vector **A** but with a magnitude that is *b* times as long as the original vector.

If, however, a vector **A** is multiplied by a negative real number *b*, then the product *b***A** is a vector with in the opposite direction to the original vector **A** but with a magnitude that is |*b*| times as long as the original vector.

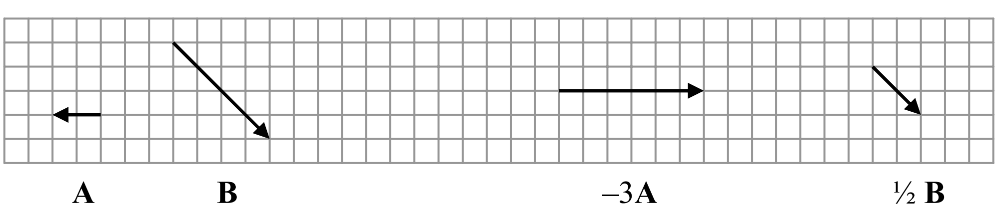
If a vector **A** is multiplied by zero, then the result is the zero vector, with 0 magnitude and no direction.

Example

For the vectors **A** and **B** sketched below, sketch –3**A** and ½ **B.**



Answer:



Example

C = 2 at 45º. Find the vectors 5**C** and –1/4 **C**.

Answer

For 5**C**, the vector is being multiplied by a positive number, so its direction remains unchanged, and the magnitude will be 5×2=10:

5**C** = 10 at 45º

For –1/4 **C**, the vector will change direction and be in QIII, so will be at an angle of (180º + 45º) = 225º. It will also only be ¼ of the original length:

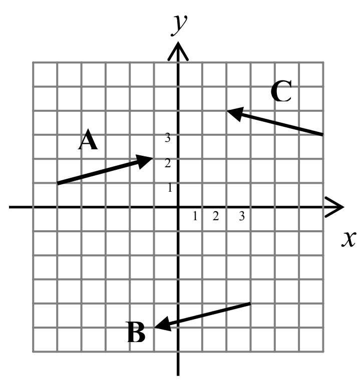
–1/4 **C** = ½ at 225º

Unit vectors

A unit vector, like the name sounds, is simply a vector that is one unit long. Such vectors are very useful for indicating directions. The unit vector in the *x*-direction has a special symbol associated with it: **i** (which unfortunately can be easily confused with the symbol used in the complex number system). Similarly, the unit vector in the *y*-direction is denoted by **j**.

Combining our ideas of unit vectors and scalar multiplication, the product 4**i** is a vector 4 units long pointing in the +*x*-direction. Let's look at an example to see why this is useful.

Let us consider our old friend vector **A**, as shown in the diagram below.



Vector **A** can be considered as the sum of a vector 4 units long in the *x*-direction plus a vector 1 unit long in the *y*-direction. Writing these two components as 4**i** and **j**, then we can write **A** as a vector sum:

$$\mathbf{A} = 4\mathbf{i} + \mathbf{j}$$

or, more generally,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$$

with A_x and A_y being the vector components we discussed before.

Example

Resolve the following vector into components. Write your answer using the **ij** notation. Give exact answers.

$\mathbf{F}$ = 35 at 225º

Answer

$$\begin{aligned}\cos\theta &= \frac{F_x}{F}\\ F_x &= F\cos\theta\\ &= 35\cos 225^\circ\\ &= -\frac{35\sqrt{2}}{2}\end{aligned}$$

and, similarly, $F_y = F\sin\theta = F\sin 35^\circ = -\dfrac{35\sqrt{2}}{2}$.

Therefore, $\mathbf{F} = -\dfrac{35\sqrt{2}}{2}\,\mathbf{i} - \dfrac{35\sqrt{2}}{2}\,\mathbf{j}$.

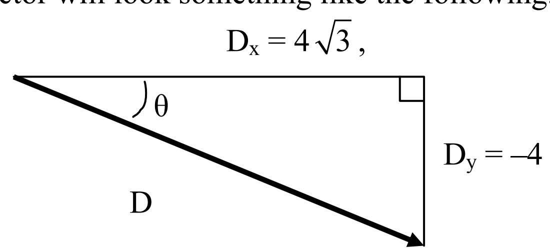
Example

Calculate the magnitude and direction of the following vector. Give exact answers.

$\mathbf{D} = 4\sqrt{3}\,\mathbf{i} - 4\,\mathbf{j}$

Answer

The x-component is positive, $4\sqrt{3}$, while the y-component, –4, is negative, so the vector will look something like the following.



We can find D, the length of the vector (and the hypotenuse of the triangle) by

$$
\begin{aligned}
c^2 &= a^2 + b^2 \\
\mathrm{D}^2 &= \mathrm{D}_x{}^2 + \mathrm{D}_y{}^2 \\
&= \left(4\sqrt{3}\right)^2 + 4^2 \\
&= 48 + 16 = 64 \\
\mathrm{D} &= 8
\end{aligned}
$$

The angle θ may be found by

$$
\begin{aligned}
\tan\theta &= \frac{\mathrm{D}_y}{\mathrm{D}_x} \\
\tan\theta &= \frac{-4}{4\sqrt{3}} \\
\theta &= \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) \\
\theta &= -30^\circ
\end{aligned}
$$

So **D** = 8 units at –30º.

Alternatively, you could notice that this is our old friend, the 30-60-90 triangle with sides in the ratio $1:\sqrt{3}:2$, and make the angle negative because it's below the *x*-axis.

Example

Resolve the following vector into components. Write your answer using the **ij** notation. Round answers to the nearest decimal place.

F = 18.2 at 107º

Answer

$$
\begin{aligned}
\cos\theta &= \frac{\mathrm{F}_x}{\mathrm{F}} \\
\mathrm{F}_x &= \mathrm{F}\cos\theta \\
&= 18.2\cos 107^\circ \\
&\cong -5.32117 \\
&\cong -5.3
\end{aligned}
$$

and, similarly, $F_y = F \sin\theta = 18.2 \sin 107^\circ \cong 17.4047 \cong 17.4$.

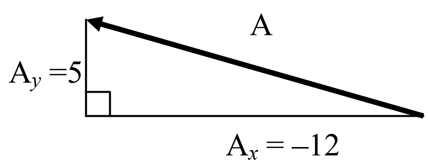
Therefore, $\mathbf{F} = -5.3\,\mathbf{i} + 17.4\,\mathbf{j}$.

Example

Find a unit vector **u** in the same direction as the vector $\mathbf{A} = -12\mathbf{i} + 5\mathbf{j}$.

Answer:

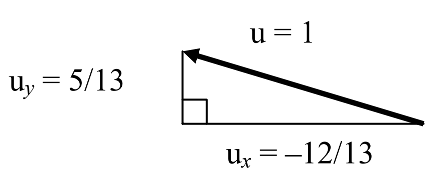
Let's first sketch vector **A**:



The length of the hypotenuse can be found by the Pythagorean theorem:

$$
\begin{aligned}
c^2 &= a^2 + b^2 \\
A^2 &= A_x{}^2 + A_y{}^2 \\
&= (-12)^2 + 5^2 \\
&= 169 \\
A &= 13
\end{aligned}
$$

The unit vector **u** will be in the same direction as **A**, but will be only one unit long. So **u** will have a triangle similar to **A**'s triangle but with a hypotenuse of 1. Therefore, if we scale **A**'s triangle down by dividing each side by 13, the length of **A**'s hypotenuse, then we'll get the similar triangle:



So $\mathbf{u} = -12/13\,\mathbf{i} + 5/13\,\mathbf{j}$.