Section V.3: Dot Product

Introduction

So far we have looked at operations on a single vector. There are a number of ways to combine two vectors. Vector addition and subtraction will not be covered here, since Physics 191 has a unit on vector addition/subtraction. We will look at the dot product, which is also known as the <u>inner</u> product or <u>scalar</u> product.

Definition

The dot product of vectors **A** and **B** is written as **A·B**. For two-dimensional vectors, the dot product is calculated by multiplying the components of **A** and **B** together as follows:

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y$$

and since A_x, A_y, B_x, and B_y are all just real numbers, the dot product is also a real number and is consequently a scalar. (This is why it's also called the <u>scalar</u> product.)

Example

Calculate the dot product **A·B** of the following vectors.

A has components $A_x = 12$, $A_y = 5$.

B has components $B_x = 2$, $B_y = –6$.

Answer:

$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y = 12\times2 + 5\times(–6) = 24 – 30 = –6$.

Example

Calculate the dot product **B·A** of the vectors in the preceding example.

Answer:

$\mathbf{B}\cdot\mathbf{A} = B_xA_x + B_yA_y = 2\times12 + (–6)\times5 = 24 – 30 = –6$.

Example

Calculate the dot product **A·B** of the following vectors.

$\mathbf{A} = 3\mathbf{i} - 5\mathbf{j}$

$\mathbf{B} = -2\mathbf{i} - 7\mathbf{j}$

Answer:

$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y = 3\times(-2) + (-5)\times(-7) = -6 + 35 = 29.$

You can see from these examples that the result of the dot product, a scalar, can be positive or negative.

You can also see that in the first two examples, $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$. Does this work for all vectors **A** and **B**? Let's look at the calculation: $A_xB_x + A_yB_y$. Since multiplication of real numbers is commutative (order doesn't matter), then $A_xB_x + A_yB_y = B_xA_x + B_yA_y$ for any vectors **A** and **B**. Therefore, $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ for all vectors **A** and **B**.

Properties of the Dot Product

We've just established that $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$. What are some other properties of the dot product? Well, $\mathbf{A}\cdot\mathbf{A} = A_xA_x + A_yA_y = (A_x)^2 + (A_y)^2 = |A|^2$, so the dot product of a vector with itself can be used to calculate the magnitude of vector **A** by taking the square root of the result. Also, when you take the dot product of the zero vector **0** and any vector, you get the scalar zero. To summarize:

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$

$$\mathbf{A}\cdot\mathbf{A} = |A|^2$$

$$\mathbf{0}\cdot\mathbf{A} = 0$$

Example

Calculate the magnitude of vector $\mathbf{V} = -24\mathbf{i} + 7\mathbf{j}$ using the dot product.

Answer:

$|V|^2 = \mathbf{V}\cdot\mathbf{V} = (V_x)^2 + (V_y)^2 = (-24)^2 + 7^2 = 576 + 49 = 625.$

Therefore, the magnitude of **V** is just the square root of 625, which is 25. Vector **V** is 25 units long.

Dot product using magnitudes and directions

The definition we've been using works very well when you have vectors **A** and **B** already in component form. However, if you instead have **A** and **B** in terms of magnitude and direction, an equivalent way of calculating the dot product is

$$\mathbf{A}\cdot\mathbf{B} = |A|\,|B| \cos\theta\ ,$$

where θ is the angle between vectors **A** and **B**. In science and engineering, we usually drop the absolute value bars and write

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta,$$

where A and B are the magnitudes of vectors **A** and **B**, respectively, and θ is once again the angle between the two vectors. You can see once again from the fact that multiplying real numbers is commutative that the order of **A** and **B** in the dot product doesn't matter.

Example

Calculate the dot product $\mathbf{A}\cdot\mathbf{B}$ of the following vectors.

$\mathbf{A}$ = 5 units at 50º

$\mathbf{B}$ = 8 units at 110º

Answer:

The angle θ between these two vectors is $(110^\circ - 50^\circ) = 60^\circ$. Since cosine is an even function, it doesn't matter which angle comes first: $\cos 60^\circ = \cos -60^\circ$. Therefore,

$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta = 5\times8\cos 60^\circ = 20.$

You can also tell from this definition that whether the dot product is positive or negative depends on the angle between the vectors. Since $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$, where A and B are the magnitudes of the vectors and are consequently always positive, the sign of $\mathbf{A}\cdot\mathbf{B}$ will depend on the properties of $\cos\theta$. If θ is acute, $\cos\theta$ will be positive. If θ is obtuse, then $\cos\theta$ will be negative. Also, there are some special cases of angles: if $\theta = 0$, $\cos\theta = 1$. If $\theta = 180^\circ$, then $\cos\theta = -1$. And lastly if $\theta = 90^\circ$ (vectors are perpendicular), then the dot product will be exactly zero.

This is an important point that bears repeating: for any vectors **A** and **B**, if **A** and **B** are perpendicular then the dot product $\mathbf{A}\cdot\mathbf{B}$ is zero. The converse is also true: if $\mathbf{A}\cdot\mathbf{B}$ is zero, then **A** and **B** are perpendicular.

It is worth noting that vectors **A** and **B** do not have to have the same units. For example, when calculating the work done by a force **F** on an object moving through displacement **d**, the work may be calculated by $W=\mathbf{F}\cdot\mathbf{D} = Fd\cos\theta$. The units of force are newtons (N), the units of displacement are metres (m) and the units of work, a scalar, are joules (J). You may recall from your previous study of physics that if the force vector is perpendicular to the displacement vector, that the force is doing no work on the object. This agrees with our observation above that if $\theta = 90^\circ$, then the dot product will be zero.

Example

Are **A** and **B** perpendicular? Use the dot product **A**·**B** to determine your answer.

A = 3**i** – 5**j**

B = –10**i** – 6**j**

Answer:

$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y = 3\times(-10) + (-5)\times(-6) = -30 + 30 = 0.$

Yes, these vectors are perpendicular since **A**·**B** = 0.

The Angle Between Two Vectors

Since **A**·**B** = AB cos θ, you can solve for cos θ to get the following equation.

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB}$$

This allows you to calculate the angle between the vectors from the magnitudes of the vectors and the dot product.

Example

Calculate the angle between the two vectors using the dot product.

A = 3**i** – 4**j**

B = –2**i** – 2**j**

Answer:

$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y = 3\times(-2) + (-4)\times(-2) = -6 + 8 = 2.$

$|A|^2 = \mathbf{A}\cdot\mathbf{A} = (A_x)^2 + (A_y)^2 = (3)^2 + (-4)^2 = 9 + 16 = 25$, so $|A| = 5$.

$|B|^2 = \mathbf{B}\cdot\mathbf{B} = (B_x)^2 + (B_y)^2 = (-2)^2 + (-2)^2 = 4 + 4 = 8$, so $|B| = \sqrt{8}$.

Therefore, $\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{AB} = \dfrac{2}{5\times\sqrt{8}} = \dfrac{2}{5\sqrt{8}}$. $\theta = \cos^{-1}\left(\dfrac{2}{5\sqrt{8}}\right) \cong 81.9^{\circ}$.

The angle between **A** and **B** is 81.9º.

Since the inverse of the cosine gives answers in quadrants I and II, you don't have to do any further calculations to get the angle. If the angle you are trying to calculate is obtuse ($90^\circ < \theta < 180^\circ$), then you'll find that the dot product is negative and taking the inverse cosine will automatically give you an answer in QII.

The Dot Product in Three Dimensions

We can generalize to three dimensions: instead of restricting ourselves to x and y, we can add a third dimension by adding a z-axis which is perpendicular to both of the x- and y-axes. This means that the z-axis would be either pointing into or out of the page that the x- and y-axes are drawn on. Our convention is that we use a right-handed system. This means that, using your right hand, if your thumb is pointing along the x-direction and your fingers are pointing along the y-direction, that your palm points in the direction of the z-axis. For our usual x-y coordinate system drawn on a sheet of paper, this means that the z-axis is pointing out of the paper towards you.

Points in this 3-dimensional space must therefore have three coordinates, not two, and are written as ordered triples: (x, y, z). Similarly, vectors will now have three components, such that vector **A** will have components A_x, A_y, and A_z. Writing in **ijk** notation, we then have **k**, the unit vector pointing along the z-direction, associated with A_z, so vector **A** could then be, for example, equal to $3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$.

Three-dimensional vectors can also be written with magnitudes and directions. However, the direction angles are more complicated than we have time for, so we will restrict ourselves to calculating the magnitude of 3D vectors. We can now state the definition of the dot product in 3D form:

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

Example

Calculate the dot product $\mathbf{A}\cdot\mathbf{B}$ of the following vectors.

$\mathbf{A} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$

$\mathbf{B} = -2\mathbf{i} - 7\mathbf{j} + \mathbf{k}$

Answer:

$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z = 3\times(-2) - 5\times(-7) + 2\times1 = -6 + 35 + 2 = 31.$

As before, the dot product may be used to find the magnitude of a 3D vector, as in the following example.

Example

Calculate the magnitude of vector **V = –4i + 7j – 3k** using the dot product.

Answer:

$|V|^2 = \mathbf{V}\cdot\mathbf{V} = (V_x)^2 + (V_y)^2 + (V_z)^2 = (-4)^2 + 7^2 + (-3)^2 = 16 + 49 + 9 = 70.$

Therefore, the magnitude of **V** is just the square root of this dot product, $\sqrt{70}$. Vector **V** is $\sqrt{70}$ units long.

We can also the dot product to find the angle between two 3D vectors in space, as in the following example.

Example

Calculate the angle between the two vectors using the dot product.

$\mathbf{A} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$

$\mathbf{B} = -2\mathbf{i} - 7\mathbf{j} + \mathbf{k}$

Answer:

$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z = 3\times(-2) - 5\times(-7) + 2\times1 = -6 + 35 + 2 = 31.$

$|A|^2 = \mathbf{A}\cdot\mathbf{A} = (A_x)^2 + (A_y)^2 + (A_z)^2 = (3)^2 + (-5)^2 + (2)^2 = 9 + 25 + 4 = 38,$
so $|A| = \sqrt{38}$.

$|B|^2 = \mathbf{B}\cdot\mathbf{B} = (B_x)^2 + (B_y)^2 + (B_z)^2 = (-2)^2 + (-7)^2 + (1)^2 = 4 + 49 + 1 = 54,$
so $|B| = \sqrt{54} = 3\sqrt{6}$.

Therefore, $\theta = \cos^{-1}\left(\dfrac{\mathbf{A}\cdot\mathbf{B}}{\mathrm{AB}}\right) = \cos^{-1}\left(\dfrac{31}{\left(\sqrt{38}\right)\left(3\sqrt{6}\right)}\right) \cong 46.8161^\circ = 46.8^\circ.$

The angle between **A** and **B** is 46.8º.