Section V.4: Cross Product

Definition

The cross product of vectors **A** and **B** is written as **A**×**B**. The result of the cross product **A**×**B** is a third vector which is perpendicular to both **A** and **B**. (Because the result is a vector, the cross product is also known as the vector product.)

In component form, the cross product is

$$\mathbf{A}\times\mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

This looks complicated and difficult to memorize. In the example below, I'll show you an easier way to calculate the cross product without memorizing the above formula.

Example

Calculate the cross product **A**×**B** for **A** = 3**i** + 5**j** – 3**k** and **B** = 2**i** – **j** + 7**k**.

Answer:

Rather than using the cumbersome formula given above, I'm going to do an easier calculation (using a method very similar to taking the determinant of a matrix, for anyone familiar with that technique):

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & -3 \\ 2 & -1 & 7 \end{vmatrix}$$

To evaluate this weird-looking thing, copy the first two columns to the right:

$$\mathbf{A}\times\mathbf{B} = \left|\begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & -3 \\ 2 & -1 & 7 \end{array}\right| \begin{array}{cc} \mathbf{i} & \mathbf{j} \\ 3 & 5 \\ 2 & -1 \end{array}$$

and draw diagonal lines connecting the columns like so:

$$\mathbf{A}\times\mathbf{B} = \left|\begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & -3 \\ 2 & -1 & 7 \end{array}\right| \begin{array}{cc} \mathbf{i} & \mathbf{j} \\ 3 & 5 \\ 2 & -1 \end{array}$$

The first diagonal line runs through the entries **i**, 5, and 7. Multiply these three together to get 35**i**. Similarly, the next two lines running in the same direction give **j**(–3)(2) = –6**j** and **k**(3)(–1) = –3**k**.

The lines running diagonally from upper right to lower left have a similar calculation, resulting in 10**k**, 3**i**, and 21**j**.

What do you do with these? The cross product is sum of all of the upper-left/lower-right diagonals <u>minus</u> the sum of all of the upper-right/lower-left diagonals.

So, $\mathbf{A}\times\mathbf{B} = (35\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}) - (10\mathbf{k} + 3\mathbf{i} + 21\mathbf{j}) = 32\mathbf{i} - 27\mathbf{j} - 13\mathbf{k}$

(It's actually easier to do the calculation than to describe it!)

Example

Calculate the cross product **B**×**A** for the same vectors as previously.

Answer:

$$\mathbf{B}\times\mathbf{A} = \left|\begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 7 \\ 3 & 5 & -3 \end{array}\right| \begin{array}{cc} \mathbf{i} & \mathbf{j} \\ 2 & -1 \\ 3 & 5 \end{array}$$

$$\mathbf{B}\times\mathbf{A} = \left|\begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 7 \\ 3 & 5 & -3 \end{array}\right| \begin{array}{cc} \mathbf{i} & \mathbf{j} \\ 2 & -1 \\ 3 & 5 \end{array}$$

$$= (3\mathbf{i} + 21\mathbf{j} + 10\mathbf{k}) - (-3\mathbf{k} + 35\mathbf{i} - 6\mathbf{j}) = -32\mathbf{i} + 27\mathbf{j} + 13\mathbf{k}$$

which is the opposite of the vector found for **A**×**B**.

We've seen for this example that **A**×**B** = – **B**×**A**. This is true for all vectors **A** and **B**, and not just for this specific case. So the <u>order</u> of vectors in the cross product matters, unlike most other multiplication you have seen, meaning that the cross product is <u>not</u> commutative.

Let's look at another example, where some (well, most!) of the vector components are zero.

Example

Calculate the cross product **A**×**B** for **A** = **j** and **B** = **i**.

Answer:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \begin{matrix} \mathbf{i} & \mathbf{j} \\ 0 & 1 \\ 1 & 0 \end{matrix} \qquad \text{(most of the diagonals will give zeros)}$$

$$= (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) - (1\mathbf{k} + 0\mathbf{i} + 0\mathbf{j}) = -\mathbf{k}$$

which is in the negative z-direction.

Let's suppose we know the magnitudes and directions of A and B instead of the component form. If **C = A×B**, then the magnitude of **C** is equal to AB sin θ, where θ is the angle between the two vectors. But what direction is **C** in? Recall that **C** is perpendicular to both **A** and **B**. It's tricky to visualize, but for any two vectors in space that aren't parallel, you can place a plane such that both vectors lie in that plane. Then **C** is perpendicular to that plane. How do you determine whether **C** points up out of the plane or down below it? You can use the right-hand rule.

The Right-Hand Rule

There are a number of different ways to determine the direction of **C** using the right-hand rule. I'll try to describe two of them, and you should probably stick with whichever one works better for you.

Method 1: Using your right hand, stick your thumb in the direction of **A**. Then, with your hand flat, stick your fingers in the direction of **B**. Your palm will then point in the direction of **C**.

Method 2: Using your right hand, flatten your hand and point your fingers in the direction of **A**. You should then rotate your hand so that your fingers curl towards **B** (and your palm is perpendicular to the **AB** plane). Then your thumb points in the direction of **C**.

The Magnitude of the Cross Product

Recall that the magnitude of vector **C** can be calculated using C = AB sin θ, where θ is the angle between vectors A and B. The geometrical interpretation is that this is the product of the vector **A** and the component of **B** which is <u>perpendicular</u> to vector **A**.

Therefore, if **A** is perpendicular to **B** (θ = 90º), then sin θ is equal to 1, and C = AB. If, however, **A** is parallel to **B** (θ = 0 or 180º), then sin θ is then zero, and C = 0, meaning that **A**×**B** = **0**.

Note that the magnitude of C is never negative. We always take the positive angle between the two vectors, so $0 \leq \theta \leq 180°$. As the sine function is + in both the first and second quadrant, we will only get zero or positive values for AB sin θ.

Example

Vector **A** is in the y-direction, while vector **B** is in the x-direction. What is the direction of **A×B? B×A?**

Answer

Using the right hand rule, Method #1, I stick my thumb along $+y$ and then twist my arm to stick my fingers along $+x$. My palm then pushes "into" the paper, which is in the $-z$-direction.

Using the right hand rule, Method #2, I stick my fingers along y, and then rotate my hand until I can curl my fingers towards $+x$. Then my thumb points "into" the page, the $-z$-direction.

(If you'll notice, this gives the same answer as the second exercise, where **A** = **j** and **B** = **i**.)

Then **B**×**A** is in the opposite direction, the $+z$-direction.

The Angle Between the Vectors

You can also use the cross product to find the angle between two vectors (though the dot product calculation is <u>much</u> easier!). Using the fact that C = AB sin θ for **C** = **A**×**B**, we get that

$$\sin\theta = \frac{\mathrm{C}}{\mathrm{AB}} = \frac{|\mathbf{A}\times\mathbf{B}|}{\mathrm{AB}} \quad .$$

Example

Use the cross product to find the angle between the following pairs of vectors: **A** = 3**i** + 5**j** – 3**k** and **B** = 2**i** – **j** + 7**k**.

Answer

First, we need to calculate the cross product **C** = **A**×**B**. I will admit to being lazy and using the same vectors as the first example, so I can use that calculation and write that **C** = **A**×**B** = 32**i** – 27**j** – 13**k**.

But we need to find C = $\left|\mathbf{A}\times\mathbf{B}\right|$. However, we know that we can calculate C either by using a 3D Pythagorus or by taking the dot product of C with itself:

$C^2 = \mathbf{C}\cdot\mathbf{C} = C_x^{\,2} + C_y^{\,2} + C_z^{\,2} = (32)^2 + (-27)^2 + (-13)^2 = 1922$

so C = $\sqrt{1922} = 31\sqrt{2}$.

Using the same technique to find A and B, we get

$A^2 = \mathbf{A}\cdot\mathbf{A} = A_x^{\,2} + A_y^{\,2} + A_z^{\,2} = (3)^2 + (5)^2 + (-3)^2 = 43$

so A = $\sqrt{43}$.

$B^2 = \mathbf{B}\cdot\mathbf{B} = B_x^{\,2} + B_y^{\,2} + B_z^{\,2} = (2)^2 + (-1)^2 + (7)^2 = 54$

so B = $\sqrt{54} = 3\sqrt{6}$.

Marching onward, we stick these quantities into our original equation:

$\sin\theta = \dfrac{C}{AB} = \dfrac{31\sqrt{2}}{\sqrt{43}\,3\sqrt{6}}$ and then solving for θ gives 65.477º, which rounds to 65º.

You will find that the cross product appears quite frequently in physics and statics, primarily through the definition of torque, $\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$, where $\boldsymbol{\tau}$ is the torque (or moment) about an axis, **r** is the moment arm or lever arm – the vector from the axis of rotation to the point where the force is being applied, and **F** is the force being applied to rotate the object. Cross products also occur when a charged particle is moving through a magnetic field ($\mathbf{F} = q\mathbf{v}\times\mathbf{B}$, where **F** is the force, q is the charge on the particle, **v** is the velocity of the particle, and **B** is the magnetic field strength) and when determining the magnetic field around a current-carrying wire. Essentially, any time you use a right-hand rule in a science or engineering course, there's usually a cross product lurking in the theory behind the phenomenon you are studying!