28.10 #20, from *Basic Technical Mathematics with Calculus* (8th Edition) by Allyn J. Washington.

$$\int \frac{-x^3 + x^2 + x + 3}{(x+1)(x^2+1)^2}\,dx$$

This example involves a repeated irreducible quadratic factor. The partial fractions set-up is

$$\frac{-x^3 + x^2 + x + 3}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}.$$

Multiplying both sides by $(x+1)(x^2+1)^2$ we obtain

$$-x^3 + x^2 + x + 3 = A(x^2+1)^2 + (Bx + C)(x+1)(x^2+1) + (Dx + E)(x+1). \qquad (1)$$

$$\text{For } x = -1, \text{ we get:} \qquad 4 = 4A + 0 + 0 \implies A = 1.$$

To obtain B, C, D, and E we will expand equation (1) and equate coefficients of powers of x.

$$\begin{aligned} -x^3 + x^2 + x + 3 &= A(x^4 + 2x^2 + 1) + (Bx + C)(x^3 + x^2 + x + 1) + (Dx + E)(x+1) \\ &= (A+B)x^4 + (B+C)x^3 + (2A + B + C + D)x^2 + (B + C + D + E)x + (A + C + E) \end{aligned}$$

By equating the corresponding coefficients of powers of x, we obtain

$$\begin{aligned} A + B &= 0 \\ B + C &= -1 \\ 2A + B + C + D &= 1 \\ B + C + D + E &= 1 \\ A + C + E &= 3. \end{aligned}$$

Since $A = 1$, we can deduce B, C, D, and E as follows.

$$\begin{aligned} A + B = 0 &\implies B = -1 \\ B + C = -1 &\implies C = 0 \\ A + C + E = 3 &\implies E = 2 \\ B + C + D + E = 1 &\implies D = 0 \end{aligned}$$

Therefore,

$$\int \frac{-x^3 + x^2 + x + 3}{(x+1)(x^2+1)^2}\,dx = \int \left(\frac{1}{x+1} - \frac{x}{x^2+1} + \frac{2}{(x^2+1)^2} \right) dx.$$

We can use the substitution $u = x^2 + 1$ to get

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C.$$

We can use the trigonometric substitution $x = \tan\theta$ to deduce that

$$\int \frac{2}{(x^2+1)^2}\,dx = \frac{x}{x^2+1} + \tan^{-1} x + C.$$

We then conclude that

$$\int \frac{-x^3 + x^2 + x + 3}{(x+1)(x^2+1)^2}\,dx = \ln|x+1| - \frac{1}{2}\ln(x^2+1) + \frac{x}{x^2+1} + \tan^{-1} x + C.$$