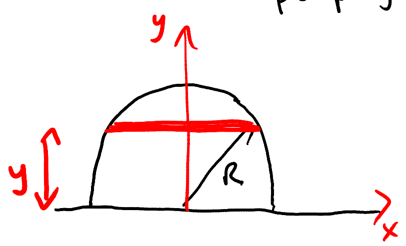
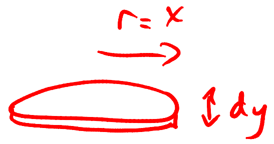
Wednesday, January 30, 2013
10:35 AM

example: A hemispherical tank of radius R is full of water. Find the work done in pumping out the tank. Assume that the pump is floating on top of the water.





disk at height y:

volume of slice

$$dV = \pi r^2 dt$$
$$= \pi x^2 dy$$

mass of slice $\quad dm = \rho \pi x^2 dy$

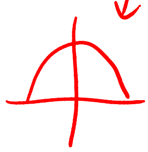
weight of slice $\quad dF_g = \rho g \pi x^2 dy$

work done in moving that slice of water to the top of the tank

$$dW = dF_g \cdot h$$
$$= \rho g \pi x^2 dy \cdot (R-y)$$

$$W = \int_h dW$$
$$= \int_0^R \rho g \pi \, x^2 (R-y)\, dy$$

what's the relation between x + y?



$x^2 + y^2 = R^2$ for circle

$x^2 = R^2 - y^2$

$$= \int_0^R \rho g \pi (R^2 - y^2)(R-y)\, dy$$

$$= \int_0^R \rho g \pi (R^3 - yR^2 - Ry^2 + y^3)\, dy$$

$$= \rho g \pi \left[R^3 y - \frac{y^2 R^2}{2} - \frac{R y^3}{3} + \frac{y^4}{4} \right] \Big|_0^R$$

$$= \rho g \pi \left[R^4 - \frac{R^4}{2} - \frac{R^4}{3} + \frac{R^4}{4} \right] - 0$$

$$= \rho g \pi R^4 \left(\frac{12}{12} - \frac{6}{12} - \frac{4}{12} + \frac{3}{12} \right)$$

$$= \frac{5}{12} \rho g \pi R^4$$

note: if $R = 3$ m and $\rho = 1000 \frac{kg}{m^3}$

then $W = \frac{5\pi}{12}\left(1000 \frac{kg}{m^3}\right)\left(9.8 \frac{N}{kg}\right)(3\ m)^4$

$= 1.04 \times 10^6$ J

$= 1$ MJ

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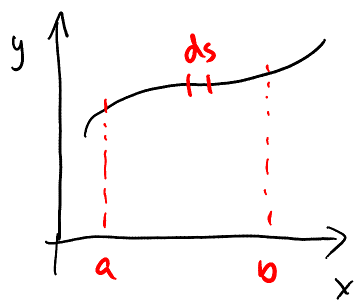
arclength:



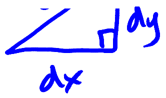
the length of the curve between a & b is

$$s = \int_a^b dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

why?

consider a little bit of arc ds

ds not exactly straight,



but as the triangle gets smaller & smaller, the approximation gets better & better

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

but, recall that $dy = \frac{dy}{dx}\,dx$

so $$ds = \sqrt{dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2}$$

$$ds = dx\,\sqrt{1 + \left(dy/dx\right)^2}$$

$$S = \int_a^b dx\,\sqrt{1 + \left(dy/dx\right)^2}$$

example: A rocket takes off on a path described by the equation

$$y = \frac{2}{3}\left(x^2 - 1\right)^{3/2}$$

Find the distance traveled by the rocket from $x = 1.0$ km to $x = 3.0$ km.

$$S = \int_a^b dx\,\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{dy}{dx} = \frac{2}{3}\,\frac{3}{2}\left(x^2 - 1\right)^{1/2} 2x$$

$$= 2x\left(x^2 - 1\right)^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = 4x^2\left(x^2 - 1\right)$$

$$= 4x^4 - 4x^2$$

$$S = \int^3 dx\,\sqrt{1 + 4x^4 - 4x^2}$$

$$S = \int_1 dx \sqrt{1 + 4x^4 - 4x^2}$$

$$= \int_1^3 dx \sqrt{4x^4 - 4x^2 + 1}$$

hint: factor inside the square root

$$= \int_1^3 dx \sqrt{(2x^2-1)^2} \quad \text{or} \quad \sqrt{(1-2x^2)^2}$$

$$= \int_1^3 dx \, (2x^2 - 1)$$

$$= \left(\frac{2}{3}x^3 - x\right)\Big|_1^3$$

$$= (18 - 3) - \left(\tfrac{2}{3} - 1\right)$$

$$= 15\tfrac{1}{3} \quad \leftarrow \text{exact answer}$$

$$= 15 \text{ km} \quad \text{(with rounding)}$$

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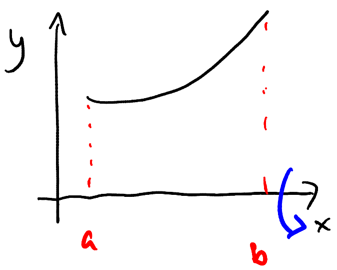
area of a surface of revolution:



← take the curve and whirl around x-axis to get a surface of revolution

→ looks like lampshade

what's the area? (don't include circular bits at the ends)

$$\boxed{\text{Surface Area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx}$$