Section 28.6: Inverse Trig Forms

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10:31 AM

Consider the integral $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$

regular substitution fails! (need an "x" in numerator)

you probably recognize this already:

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

why is this true? we need a different type of substitution: (look at this method in section 28.8)

$$\int \frac{dx}{\sqrt{1-x^2}}$$

let $\sin u = x$
$\cos u \, du = dx$ } Section 28.8

if $\sin u = x$
$u = \sin^{-1} x$

$$= \int \frac{\cos u \, du}{\sqrt{1-\sin^2 u}}$$

$$= \int \frac{\cos u \, du}{\sqrt{\cos^2 u}}$$

$$= \int \frac{\cos u \, du}{\cos u}$$

$$= \int du$$

$$= u + C$$

$$= \sin^{-1} x + C$$

in general: $$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$ on formula sheet

similarly $$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

examples:

warmups:

$$\int \frac{dx}{\sqrt{16-x^2}} = \sin^{-1}\left(\frac{x}{4}\right) + C$$

$$\int \frac{dx}{x^2+5} = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

$$= \frac{\sqrt{5}}{5}\tan^{-1}\left(\frac{\sqrt{5}\,x}{5}\right) + C$$

examples:

$$\int \frac{dx}{x^2+8x+25}$$

$$= \int \frac{dx}{x^2+8x+\underline{16}+25-\underline{16}}$$

$$= \int \frac{dx}{(x+4)^2+9}$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x+4}{3}\right) + C$$

$$\int \frac{dx}{\sqrt{6x - x^2}} = \int \frac{dx}{\sqrt{9 - (x^2 - 6x + 9)}}$$

$$= \int \frac{dx}{\sqrt{9 - (x-3)^2}}$$

$$= \sin^{-1}\left(\frac{x-3}{3}\right) + C$$

more examples:

$$\int \frac{5p^2}{9 + p^6}\, dp \qquad \text{let } u = p^3, \quad du = 3p^2\,dp$$

$$= \int \frac{5}{3} \frac{du}{9 + u^2}$$

$$= \frac{5}{3} \frac{1}{3} \tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{5}{9} \tan^{-1}\left(\frac{p^3}{3}\right) + C$$

Compare and contrast:

$$\int \frac{x\,dx}{x^2+1} \quad \text{vs} \quad \int \frac{dx}{x^2+1} \quad \text{vs} \quad \int \frac{x+1}{x^2+1}\,dx$$

substitution
let $u = x^2+1$

arctan

break into

$$\int \left(\frac{x}{x^2+1} + \frac{1}{x^2+1}\right) dx$$

$$\int \frac{e^{3x}}{4+e^{6x}}\,dx \quad \text{vs} \quad \int \frac{e^{3x}}{4+e^{3x}}\,dx$$

let $u = e^{3x}$
$du = 3e^{3x}\,dx$

$$= \int \frac{1}{3}\frac{du}{4+u^2}$$

$$= \frac{1}{3}\,\frac{1}{2}\tan^{-1}\frac{u}{2} + C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{e^{3x}}{2}\right) + C$$

let $u = e^{3x}+4$
$du = 3e^{3x}\,dx$

$$= \int \frac{1}{3}\frac{du}{u}$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln\left(e^{3x}+4\right) + C$$

example:

$$\int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{4-\tan^2\theta}}\,d\theta$$

let $u = \tan\theta$
$du = \sec^2\theta\,d\theta$

when $\theta = 0$, $\tan\theta = 0$
$\theta = \pi/4$, $\tan\theta = 1$

$$= \int_0^1 \frac{du}{\sqrt{4-u^2}}$$

$$= \sin^{-1}\left(\frac{u}{2}\right)\Big|_0^1$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}0$$

$$= \frac{\pi}{6}$$