Section 28.9: Integration by Partial Fractions: Nonrepeated Linear Factors

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10:47 AM

consider $\int \frac{1}{x^2-5x+6}\,dx$

→ you could complete the square and then use trig substitution → awful!

or if you knew that

$$\frac{1}{x^2-5x+6} = \frac{1}{x-3} - \frac{1}{x-2}$$

then $\int \frac{1}{x^2-5x+6}\,dx = \int \frac{1}{x-3}\,dx - \int \frac{1}{x-2}\,dx$

$$= \ln|x-3| - \ln|x-2| + C$$

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so, how do you break apart the fraction $\frac{1}{x^2-5x+6}$ in bits (partial fractions)

consider $\frac{x-7}{x^2+6x+5} = \frac{x-7}{(x+1)(x+5)} = \frac{A}{x+1} + \frac{B}{x+5}$

↑ ↑
linear factors
& they are not the same
(non-repeating)

so how do you find A & B?

multiply both sides by $(x+1)(x+5)$

$$(x+1)(x+5)\left(\frac{x-7}{x^2+6x+5}\right) = \left(\frac{A}{x+1} + \frac{B}{x+5}\right)(x+1)(x+5)$$

$$x-7 = A(x+5) + B(x+1)$$

method #1: long and irritating! DO NOT USE!

$$x-7 = Ax + Bx + 5A + B$$
$$x-7 = (A+B)x + (5A+B)$$

system of equations: $A+B=1$, $5A+B=-7$

$\to$ solve it!

method #2: USE THIS ONE!

$$x-7 = A(x+5) + B(x+1)$$

$\uparrow$ choose $x=-5$

$$-12 = A\cdot 0 + B(-4)$$
$$B = 3$$

now choose $x=-1$

$$-8 = 4A$$
$$A = -2$$

then

$$\int \frac{x-7}{x^2+6x+5}\,dx = \int \left[\frac{-2}{x+1} + \frac{3}{x+5}\right] dx$$

$$= -2\ln|x+1| + 3\ln|x+5| + C$$

note: this expansion: $\dfrac{A}{x-d} + \dfrac{B}{x-f} + \dfrac{C}{x-g}$

only works when

1. each factor in the denominator is raised only to the first power
2. each denominator is linear (degree is 1 - the exponent on x is 1)
3. for the fraction, the degree of the numerator < degree of the denominator

→ 1 + 2: if these aren't true, use techniques of next section

3: you're not going to like the answer!

$$\int \frac{x^3 - x + 3}{x^2 + x - 2}\,dx$$

← degree of numerator ≥ degree of denominator

long division!

$$
\begin{array}{r|rrrr}
 & x & -1 & & \\
\hline
x^2 + x - 2 & x^3 & +\,0x^2 & -\,x & +\,3 \\
 & x^3 & +\,x^2 & -\,2x & \\
\hline
 & & -x^2 & +\,x & +\,3 \\
 & & -x^2 & -\,x & +\,2 \\
\hline
 & & & 2x & +\,1
\end{array}
$$

so $\displaystyle\int \frac{x^3 - x - 3}{x^2 + x - 2} = \int \left[x - 1 + \frac{2x+1}{x^2+x-2} \right] dx$

partial fractions :

$$\frac{2x+1}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$2x+1 = A(x-1) + B(x+2)$$

set $x=1$ $\quad 3 = 3B \quad$ so $B=1$

$x=-2$ $\quad -3 = -3A \quad A=1$

$$\text{so} \int \frac{x^3-x+3}{x^2+x-2}\,dx = \int \left[x-1 + \frac{1}{x+2} + \frac{1}{x-1} \right] dx$$

$$= \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C$$