Section 29.3: Partial Derivatives

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10:27 AM

We've been doing "calculus of a single variable" up until now

What happens when we try to take the derivative of a function of two or more variables?

→ these derivatives are called "partial derivatives" and are written:

$$\frac{\partial f}{\partial x}$$

∂ ← like a backwards six – not a "dee"

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

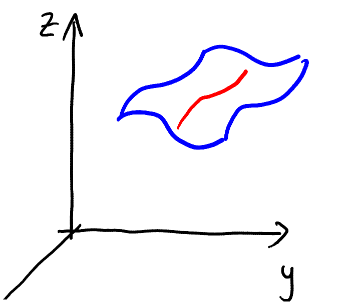
other notations

$$f_x(x,y)\ ,\quad f_y(x,y)$$

$$\frac{\partial}{\partial x} f(x,y)\ ,\quad \frac{\partial}{\partial y} f(x,y)$$

note: must specify which variable you are differentiating with respect to!

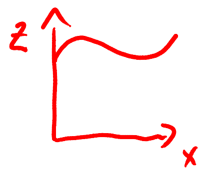
main idea:



← some curved surface $f(x,y)$

- take a slice parallel

to the x-z plane



← the slope of this line is $\frac{\partial f}{\partial x}$

so, how do you calculate partial derivatives?

- treat the other variable as if it were a constant
  → differentiate as usual

example: find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = x^2 + 2xy$

$$\frac{\partial f}{\partial x} = 2x + 2y$$

$$\frac{\partial f}{\partial y} = 0 + 2x = 2x$$

note: the actual full definition of partial derivative:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

but we won't get into that

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examples:

find $\left.\frac{\partial z}{\partial y}\right|_{(2, \pi/2, 4)}$ for $z = x^2 \cos 4y$

$$\frac{\partial z}{\partial y} = -4x^2 \sin 4y$$

$$\left.\frac{\partial z}{\partial y}\right|_{(2, \pi/2, 4)} = -4(2)^2 \sin\left(4 \cdot \pi/2\right)$$

$$= 0$$

let $z = e^y \ln x$

find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, and $\frac{\partial^2 z}{\partial y \partial x}$

so what is $\frac{\partial^2 z}{\partial x^2}$? it's what you get when you differentiate z with respect to x twice

so $z = e^y \ln x$

$$\frac{\partial z}{\partial x} = \frac{e^y}{x}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{e^y}{x^2}$$

note: $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

so $z = e^y \ln x$

$$\frac{\partial z}{\partial y} = e^y \ln x \leftarrow$$

$$\frac{\partial^2 z}{\partial y^2} = e^y \ln x$$

now, let's do $\frac{\partial^2 z}{\partial x \partial y}$:

$e^y \ln x$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(e^y \ln x\right)$$

$$= \frac{e^y}{x}$$

$\frac{e^y}{x}$

and $\frac{\partial^2 z}{\partial y \partial x}$:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{e^y}{x}\right)$$

$$= \frac{e^y}{x}$$

but, notice that $\underbrace{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$

always true for all $z(x,y)$
<u>if</u> z is continuous

so, there are 4 second derivatives in all:

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \underbrace{\frac{\partial^2 z}{\partial x \partial y}, \text{ and } \frac{\partial^2 z}{\partial y \partial x}}$$

same for nice,
continuous functions

one last example:

let $z = y^2 \sin 2x$

find all four second derivatives

$$\frac{\partial z}{\partial x} = 2y^2 \cos 2x \quad , \quad \frac{\partial z}{\partial y} = 2y \sin 2x$$

$$\frac{\partial^2 z}{\partial x^2} = -4y^2 \sin 2x \quad , \quad \frac{\partial^2 z}{\partial y^2} = 2 \sin 2x$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(2y^2 \cos 2x\right) \quad , \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(2y \sin 2x\right)$$

$$= 4y \cos 2x \qquad\qquad = 4y \cos 2x$$

same!