Section 29.4: Double Integrals

Thursday, February 28, 2013
10:32 AM

So, what about integration of functions of two variables?

double integral:

$$\underbrace{\int_a^b \underbrace{\int_{g(x)}^{G(x)} f(x,y)\, dy}_{y \text{ runs from } g(x) \text{ to } G(x)}\, dx}_{x \text{ runs from } a \text{ to } b}$$

how to deal? do inside integral first!

example: evaluate

$$\int_0^5 \int_0^{x^3} (x+y)\, dy\, dx$$

$$\int_0^5 \underbrace{\int_0^{x^3} (x+y)\, dy}_{\text{do this first}}\, dx = \int_0^5 \left[\left(xy + \frac{y^2}{2} \right) \Big|_0^{x^3} \right] dx$$

$$= \int_0^5 \left[x \cdot x^3 + \frac{x^6}{2} - 0 \right] dx$$

$$= \left(\frac{x^5}{5} + \frac{x^7}{14} \right) \Big|_0^5$$

$$= 5^4 + 5^7$$

$$\overline{14}$$

$$= \frac{86875}{14}$$

$$\approx 6205.36$$

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example: evaluate:

pi $\rightarrow$ $$\int_0^{\pi} \underbrace{\int_1^3 x \sin y \, dx}\, dy$$

$$= \int_0^{\pi} \left[\frac{x^2}{2} \sin y \Big|_1^3 \right] dy$$

$$= \int_0^{\pi} \left[\frac{9}{2} \sin y - \frac{1}{2} \sin y \right] dy$$

$$= \int_0^{\pi} 4 \sin y \, dy$$

$$= -4 \cos y \Big|_0^{\pi}$$

$$= -4 \cos \pi + 4 \cos 0$$

$$= -4(-1) + 4(1)$$

$$= 8$$

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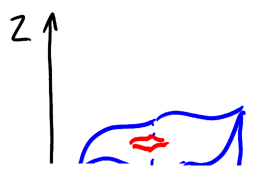
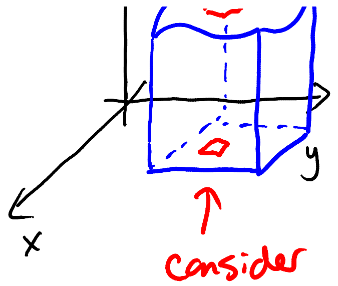
applications: volume under a 3D surface

consider a surface $z = f(x, y)$

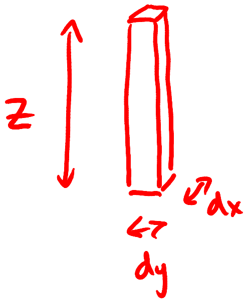


$\leftarrow$ 3D surface $\leftarrow$ the volume underneath



consider the little area element dA in the x-y plane

this surface can be found using a double integral

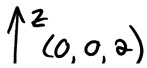


$$dV = z\,dA$$
$$= z\,dx\,dy$$
$$= f(x,y)\,dx\,dy \qquad (\text{or } f(x,y)\,dy\,dx)$$

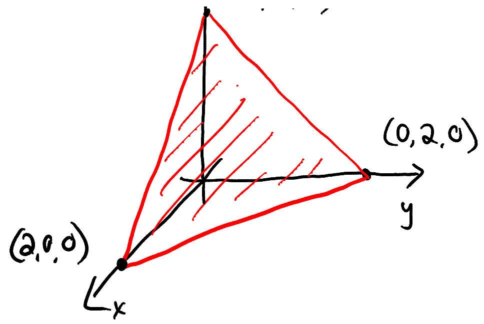
in general:

$$V = \int_V dV$$
$$= \int_V f(x,y)\,dA$$
$$= \int_V f(x,y)\,\underbrace{dx\,dy}_{\text{or } dy\,dx}$$

lets do an example, including how to set up the limits

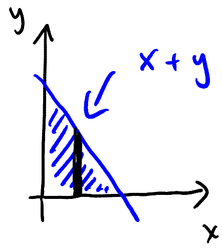


Find the volume under



Find the volume under the plane $x+y+z=2$ in the first octant.

Step 1: draw a picture of the x-y plane



$x+y=2$ (note: $z=0$)

note: $y=2-x$

step 2: take a slice to set up the limits

hard limits: $0 \le x \le 2$

soft limits $0 \le y \le 2-x$

top of slice

step 3: set up integral:

$$V = \int_V f(x,y)\, dA$$

$$= \int_0^2 \int_0^{2-x} f(x,y)\, dy\, dx$$

but what's this? it's z

$$x+y+z=2$$

$$z = 2-x-y$$

$$= \int_0^2 \int_0^{2-x} (2-x-y)\, dy\, dx$$

$$= \int_0^2 \left(2y - xy - \frac{y^2}{2}\right)\Big|^{2-x} dx$$

$$\int_0 \int_0$$

$$= \int_0^2 \left(2y - xy - \frac{y^2}{2}\right)\Big|_0^{2-x} \, dx$$

$$= \int_0^2 \left[2(2-x) - x(2-x) - \frac{(2-x)^2}{2}\right] dx$$

$$= \int_0^2 \left[(2-x)^2 - \frac{(2-x)^2}{2}\right] dx$$

$$= \int_0^2 \frac{(2-x)^2}{2} \, dx$$

$$= -\frac{(2-x)^3}{6}\Big|_0^2$$

$$= 0 + \frac{2^3}{6}$$

$$= 4/3$$