Sections 30.2/30.3/30.4: cont'd:

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10:36 AM

wrapping up Maclaurin series:

why does it work? (will not test this)

the big idea:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$f(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0 + \ldots$$
$$= a_0$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

$$f'(0) = a_1$$

$$f''(x) = 2a_2 + 3 \cdot 2 \cdot a_3 x + \ldots$$

$$f''(0) = 2a_2 \qquad \text{so} \qquad a_2 = \frac{f''(0)}{2}$$

$$f'''(x) = \underbrace{3 \cdot 2}_{} \cdot a_3$$

this is where the factorials come in

$$3 \cdot 2 \cdot 1 = 3!$$

binomial expansion:

$f(x) = (1+x)^k$	$f(0) = 1$
$f'(x) = k(1+x)^{k-1}$	$f'(0) = k$
$f''(x) = k(k-1)(1+x)^{k-2}$	$f''(0) = k(k-1)$
$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$	$f'''(0) = k(k-1)(k-2)$

$$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \quad \Big| \quad f'''(0) = k(k-1)(k-2)$$

so the definition is

$$f(x) = f(0) + \frac{f'(0)\,x}{1!} + \frac{f''(0)\,x^2}{2!} + \frac{f'''(0)\,x^3}{3!} + \cdots$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

note: $k(k-1)(k-2) = \dfrac{k!}{(k-3)!}$

NOTE: k doesn't have to be an integer!

$$\sqrt{1+x} = (1+x)^{1/2}$$

$$= 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \cdots$$

$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \cdots$$

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example: Use the first three terms of the binomial expansion to estimate $(1.006)^8$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots$$

$$(1+0.006)^8 \approx 1 + 8(0.006) + \frac{8\cdot 7}{2}(0.006)^2$$

$$\approx 1 + 0.048 + 0.001008$$

$$\approx 1.04901$$

calculator says 1.04902

example: Use the first three terms of an appropriate series expansion to estimate

$$\int_0^{0.1} \sqrt[3]{1+3x^2}\, dx$$

$$(1+x)^k \approx 1 + kx + \frac{k(k-1)x^2}{2!}$$

$$(1+3x^2)^{1/3} \approx 1 + \tfrac{1}{3}(3x^2) + \frac{\tfrac{1}{3}(-\tfrac{2}{3})}{2}(3x^2)^2$$

$$\approx 1 + x^2 - x^4$$

$$\int_0^{0.1} \sqrt[3]{1+3x^2}\, dx \approx \int_0^{0.1} (1 + x^2 - x^4)\, dx$$

$$\approx \left(x + \frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^{0.1}$$

$$\approx 0.1 + \frac{(0.1)^3}{3} - \frac{(0.1)^5}{5}$$

$$\approx 0.100331$$

Ti-89 says 0.100331

Wow!

example: Using the definition of Maclaurin series, find the first four non-zero terms of the Maclaurin series for

$$f(x) = \frac{1}{}$$

$$1 - x$$

$$f(x) = (1-x)^{-1} \qquad f(0) = 1$$

$$f'(x) = +(1-x)^{-2} \qquad f'(0) = 1$$

$$f''(x) = +2(1-x)^{-3} \qquad f''(0) = 2$$

$$f'''(x) = +6(1-x)^{-4} \qquad f'''(0) = 6$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)x^3}{2!} + \frac{f'''(0)\,x^3}{3!} + \cdots$$

$$(1-x)^{-1} \approx 1 + x + x^2 + x^3$$