**Math 189 – Assignment #1** **Name:** Solution Set

Total: 30

1. State the order and degree of the following differential equations.

a) $x^4(y''')^2 - x^3y'' + (y')^5 = y^4$

3rd order
2nd degree

(1)

b) $\dfrac{d^2x}{dt^2} = \dfrac{dx}{dt} - 5xe^{-2t}$

2nd order
1st degree

(1)

2. Show that $x^2 + y^2 = cx$ is a solution to the following DE.

$$2xyy' + x^2 = y^2$$

(-½) if forgot ±

method #1: implicit

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(cx)$

$2x + 2yy' = c$

so $2yy' = c - 2x$

$2xyy' = cx - 2x^2$

now sub $2xyy'$ into DE

$cx - 2x^2 + x^2 = y^2$

$cx - x^2 = y^2$

but from soln $x^2 + y^2 = cx$

$y^2 = cx - x^2$

so

$cx - x^2 = cx - x^2$ ✓

QED

method #2: explicit

$x^2 + y^2 = cx$

$y = \pm\sqrt{cx - x^2}$

$y' = \pm\frac{1}{2}\frac{1}{\sqrt{cx - x^2}} \cdot (c - 2x)$

$= \pm\frac{c - 2x}{2\sqrt{cx - x^2}}$

DE:

$2xyy' + x^2 = y^2$

$\cancel{2}x\left(\pm\cancel{\sqrt{cx - x^2}}\right)\left(\pm\frac{c - 2x}{\cancel{2}\cancel{\sqrt{cx - x^2}}}\right) + x^2 = cx - x^2$

$cx - 2x^2 + x^2 = cx - x^2$

$cx - x^2 = cx - x^2$ ✓

QED

(3)

3. Solve the following DE. You may leave your answer in implicit form.

$$\sqrt{1+4x^2}\,dy = y^3 x dx$$

$$\frac{dy}{y^3} = \frac{x\,dx}{\sqrt{1+4x^2}} \quad \text{separable}$$

let $u = 1+4x^2$
$du = 8x\,dx$

$$\frac{dy}{y^3} = \frac{1}{8}\frac{du}{\sqrt{u}}$$

$$\int y^{-3}\,dy = \int \frac{1}{8} u^{-1/2}\,du$$

$$\frac{y^{-2}}{2} = \frac{2}{8}u^{1/2} + C$$

$$\boxed{-\frac{1}{2y^2} = \frac{1}{4}\sqrt{1+4x^2} + C}$$

or

$$\boxed{\frac{-2}{y^2} = \sqrt{1+4x^2} + C}$$

or

$$\boxed{\sqrt{1+4x^2} + \frac{2}{y^2} = C}$$

etc

3

4. Solve, giving an explicit solution.

$$e^{x+y}(dx+dy) + 4xdx = 0$$

let $u = x+y$
$du = dx+dy$

separable $\rightarrow$ $e^u\,du + 4x\,dx = 0$

$$\int e^u\,du + \int 4x\,dx = C$$

$$e^u + 2x^2 = C$$

$$e^{x+y} + 2x^2 = C$$

$$e^{x+y} = C - 2x^2$$

$$x+y = \ln(C-2x^2)$$

$$\boxed{y = \ln(C-2x^2) - x}$$

3

5. Solve the following linear differential equation under the given conditions. Give an explicit answer.

$$\frac{dv}{dt} - \frac{v}{t} = \ln t \qquad \text{given that when } t = 1,\ v = 8.$$

linear with integrating factor

$$e^{\int P(t)dt} = e^{\int -\frac{1}{t}dt}$$

$$= e^{-\ln t} = e^{\ln 1/t} = 1/t$$

(1)

then multiply both sides:

$$\frac{dv}{dt}\cdot\frac{1}{t} - \frac{v}{t^2} = \frac{\ln t}{t}$$

$$\frac{d}{dt}\left(v\cdot\frac{1}{t}\right) = \frac{\ln t}{t}$$

(5)

$$v\cdot\frac{1}{t} = \int \frac{\ln t}{t}\,dt$$ (2)

let $u = \ln t$

$du = \frac{1}{t}dt$

$$= \int u\,du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{(\ln t)^2}{2} + C$$

so $v = \frac{t(\ln t)^2}{2} + Ct$ (1)

$$\boxed{v = \frac{t(\ln t)^2}{2} + 8t}$$

(1)

but $v(1) = 8$

$8 = 0 + C$ (since $\ln 1 = 0$)

$C = 8$

6. Consider the following differential equation:

$$\frac{dy}{dx} = \sqrt{1+xy}$$

a) Use Euler's method to approximate the solution to the following differential equation for $x = 0$ to $x = 0.2$ using a step size of $\Delta x = 0.05$. The point (0,1) lies upon the curve.

| x | y | f(x, y) | y + f(x, y) · Δx |
|---|---|---|---|
| 0 | 1 | 1 | 1.05 |
| 0.05 | 1.05 | 1.02591 | 1.1013 |
| 0.1 | 1.1013 | 1.05363 | 1.153977 |
| 0.15 | 1.153977 | 1.083096 | 1.208132 |
| 0.2 | 1.208132 | | |

columns of interest

3

b) If you were not impressed with the accuracy of your calculation in part (a) but wanted to keep using the same method, what might you change to improve the accuracy?

decrease the step size

1

c) If you really wanted to do a good job, what method would you use instead? What is it about the Euler method that makes this second method a better choice? (Please explain, but be brief!)

Euler uses the slope at the left-hand side of the interval. Runge Kutta averages the slopes over the interval instead.

2

Heun's method (improved Euler) averages the slope at the LHS and an estimate of the slope at the RHS.

7. The magnitude of the velocity $v$ of a meteor approaching the earth is given by

$$v\frac{dv}{dr} = -\frac{GM}{r^2}$$

where $r$ is the distance from the centre of the earth, $M$ is the mass of the earth, and G is the universal gravitational constant. If $v = 0$ for $r = r_o$, solve for $v$ as a function of $r$.

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$$v\frac{dv}{dr} = -\frac{GM}{r^2}$$ ⇐ seperable

$$\int v\,dv = \int -\frac{GM\,dr}{r^2}$$

$$\frac{v^2}{2} = +\frac{GM}{r} + C$$

4

at $r = r_o$, $v = 0$

$$0 = \frac{GM}{r_o} + C$$

$$C = -\frac{GM}{r_o}$$

$$v = \sqrt{2GM\left(\tfrac{1}{r} - \tfrac{1}{r_o}\right)}$$

so $$\frac{v^2}{2} = \frac{GM}{r} - \frac{GM}{r_o}$$

$$= GM\left(\frac{1}{r} - \frac{1}{r_o}\right)$$

$$v = \pm\sqrt{2GM\left(1/r - 1/r_o\right)}$$

↑

but magnitude is always $\geq 0$

8. Consider the differential equation $y'' + 6y' + ky = 0$.

Solve it for (a) $k = 8$ (b) $k = 9$ (c) $k = 10$.

auxiliary equation: $m^2 + 6m + k = 0$

for simplicity, I'll use quadratic formula for all cases subbing in for k:

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{36 - 4k}}{2}$$

When $k = 8$, $m = \frac{-6 \pm 2}{2} = -4, -2$

4

$k = 9$, $m = \frac{-6 \pm 0}{2} = -3$ (repeated)

$k = 10$, $m = \frac{-6 \pm 2i}{2} = -3 \pm i$

$m = \alpha \pm \beta i$
$\alpha = -3$
$\beta = 1$

a) $y = C_1 e^{-4x} + C_2 e^{-2x}$

b) $y = (C_1 x + C_2) e^{-3x}$

c) $y = e^{\alpha x}(C_1 \sin \beta x + C_2 \sin \beta x)$

$= e^{-3x}(C_1 \sin x + C_2 \cos x)$