Math 189 – Helpful Hints for Linear DEs

Linear DEs can be written in the form:

$$\frac{dy}{dx} + P(x)\ y = Q(x)$$

Here's what to do:

1. If your DE isn't already in exactly the form above, rewrite it so that it is.
2. Calculate the integrating factor: $e^{\int P(x)dx}$. Be sure to include any – signs in $P(x)$.
3. Multiply both sides (all terms) by the integrating factor.
4. Notice that the left-hand-side is always equal to d(y × integrating factor).
5. Integrate both sides. After integrating the left-hand-side will always be equal to y × integrating factor, so the only thing that requires care is integrating the right-hand-side.
6. Tidy up your answer. If there are initial values (such as "v = 0 when t = 0"), use them to calculate any constants. If you are required to solve for a particular variable, then do so.
7. Ta da!

Example 1: Solve the linear DE: $2dy - 5dx = -6ydx$

I will solve this using the steps labelled above.

1. Rewriting:

$$2dy - 5dx = -6ydx$$
$$2\frac{dy}{dx} - 5 = -6y$$
$$2\frac{dy}{dx} + 6y = 5$$
$$\frac{dy}{dx} + 3y = \frac{5}{2}$$

so $P(x) = 3$. (Note that $P(x)$ is "whatever is multiplying the linear (y) term" and must include negative signs!).

2. Calculate the integrating factor: $e^{\int P(x)dx} = e^{\int 3dx} = e^{3x}$

3. Multiply both sides of DE by integrating factor: $e^{3x}\dfrac{dy}{dx} + 3e^{3x}y = \dfrac{5}{2}e^{3x}$

4. Notice that the left-hand-side is just $d\left(y\,e^{3x}\right)$ so DE is now $d\left(y\,e^{3x}\right) = \dfrac{5}{2}e^{3x}$.

5. Integrate both sides, recalling that the left-hand-side will become y times the integrating factor:

$$\begin{aligned} \int d\left(y\,e^{3x}\right) &= \int \frac{5}{2}e^{3x}dx \\ ye^{3x} &= \frac{5}{6}e^{3x} + C \end{aligned}.$$

6. Tidy up: $y = \dfrac{5}{6} + Ce^{-3x}$ or, if you prefer, $y = \dfrac{5}{6} + \dfrac{C}{e^{3x}}$.

Example 2:

Solve the linear DE: $y' + y\tan x = -\sin x$

1. This one is already in the correct form, with $P(x) = \tan x$.

2. Calculate the integrating factor: $e^{\int P(x)dx} = e^{\int \tan x\,dx} = e^{-\ln(\cos x)} = e^{\ln(\cos x)^{-1}} = \dfrac{1}{\cos x} = \sec x$

3. Multiply both sides of DE by the integrating factor:

$$\sec x\; y' + y\tan x \sec x = -\sin x\left(\sec x\right)$$

4. Notice that the left-hand-side is just $d\left(y\;\sec x\right)$ so DE is now

$$d\left(y\sec x\right) = -\tan x.$$

5. Integrate both sides, recalling that the left-hand-side will become y times the integrating factor:

$$\begin{aligned} \int d\left(y\sec x\right) &= -\int \tan x\,dx \\ y\sec x &= -\ln\left(\sec x\right) + C \end{aligned}$$

6. Tidy up:

$$\begin{aligned} y\sec x &= \ln\left(\sec x\right)^{-1} + C \\ y\sec x &= \ln\left(\cos x\right) + C \end{aligned}$$