Section 31.2: Separation of Variables

Wednesday, April 03, 2013
11:38 AM

so we'll now learn some methods for how to solve DEs

first up: separation of variables

consider a DE of first order and first degree

$$\frac{dy}{dx} = f(x, y)$$

you can always rewrite this into differential form:

$$\underbrace{M(x,y)}_{\text{some function}}\,dx + \underbrace{N(x,y)}_{\text{some other function}}\,dy = 0$$

IF we can rewrite this into:

not always possible!

$$\underbrace{A(x)\,dx}_{\text{contains only } x} + \underbrace{B(y)\,dy}_{\text{contains only } y} = 0$$

then we just integrate to solve.

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example: solve

$$2\frac{dy}{dx} = \frac{y(x+1)}{x}$$

$$2\,dy = y\frac{(x+1)}{x}\,dx$$

$$\frac{2\,dy}{y} = \frac{x+1}{x}\,dx$$

← variables now separated

$$\int \frac{2\,dy}{y} = \int \frac{x+1}{x}\,dx$$

$$2\ln|y| = \int \left(1 + \frac{1}{x}\right) dx$$

note: text isn't very strict about the absolute values

$$= x + \ln x + C$$

note: already dropped abs value signs

the solution to an DE is

$$2\ln y = x + \ln x + C$$

note: perfectly acceptable answer <u>even</u> though it's not solved for y

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example: solve

$$y\,dt + t\,dy = 0$$

$$\frac{y\,dt}{yt} + \frac{t\,dy}{yt} = \frac{0}{yt}$$

$$\frac{dt}{t} + \frac{dy}{y} = 0$$

$$\int \frac{dt}{t} + \int \frac{dy}{y} = \int 0$$

perfecty acceptable answer #1 $\rightarrow$

$$\ln t + \ln y = C$$

$$\ln ty = C$$

$$ty = e^{C}$$

what's a constant raised to a constant?

perfectly acceptable answer #2 $\rightarrow$

$$ty = C^{*}$$

$\uparrow$ a different constant!

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example:

solve the following DE, given that $y = \pi/2$ when $x = 0$

$$2y\cos y\,dy - \sin y\,dy = y\sin y\,dx$$

$$2\cot y\,dy - \frac{dy}{y} = dx$$

$$\left(2\cot y - \frac{1}{y}\right)dy = dx$$

$$\int\left(2\cot y - \frac{1}{y}\right)dy = \int dx$$

$$2\ln|\sin y| - \ln y = x + C$$

$\in$ general solution

but when $x = 0$. $y = \pi/2$

but when $x=0$, $y=\pi/2$

$$2\ln\left|\sin \pi/2\right| - \ln \pi/2 = 0 + C$$

$$2\,\overset{0}{\cancel{\ln 1}} - \ln \pi/2 = C$$

$$C = -\ln \pi/2$$

so $2\ln|\sin y| - \ln y = x - \ln \pi/2$

perfecty acceptable answer #1

$$x = 2\ln|\sin y| - \ln y + \ln \pi/2$$

$$= \ln \sin^2 y - \ln y + \ln \pi/2$$

$$= \ln \frac{\pi \sin^2 y}{2y}$$

$\leftarrow$ particular solution

perfectly acceptable answer #2