Section 31.4: Linear DEs of the First order

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1:47 PM

Suppose you are able to rewrite a DE into the form

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

functions of $\underline{\underline{x}}$ $\underline{\underline{\text{only}}}$

this DE is called a <u>linear DE of first order</u>

how to solve it?

1. multiply both sides by the "integrating factor"

$$e^{\int P(x)\,dx}$$

note: include negative signs!

2. notice that the LHS will be

$$d\left(y\, e^{\int P(x)\,dx}\right)$$

3. integrate both sides

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example: which of the following are linear and of first order?

a) $y' + \frac{y}{x} = x^3$ ✓

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b) $\frac{dy}{dx} + y^2 = e^x$

c) $x\,dy + 2y\,dx = 3x\,dx$

$$x\frac{dy}{dx} + 2y = 3x$$

$$\frac{dy}{dx} + \frac{2}{x}y = 3$$

d) $yy' + 4 = \sin x$

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skill-builder:

$$\frac{d}{dx}\left(ye^x\right) = ye^x + \frac{dy}{dx}e^x$$

$$\frac{d}{dx}\left(yx^3\right) = \frac{dy}{dx}x^3 + y\cdot 3x^2$$

$$= \frac{dy}{dx}x^3 + 3x^2y$$

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full example:

solve the following DE:

$$dy - 3y\,dx = e^{3x}\,dx$$

write in derivative form:

write in derivative form:

$$\frac{dy}{dx} - 3y = e^{3x}$$

note: pattern for linear DE:

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

this is linear with

$$P(x) = -3$$

$$Q(x) = e^{3x}$$

now, figure out the integrating factor:

$$e^{\int P(x)dx} = e^{\int -3dx}$$

$$= e^{-3x}$$

note: omit $+C$

then, multiply both sides of DE by the integrating factor:

$$\frac{dy}{dx} - 3y = e^{3x}$$

$$\underbrace{\frac{dy}{dx}e^{-3x} - 3y\,e^{-3x}} = e^{3x}e^{-3x}$$

$$\frac{d}{dx}\left(y\,e^{-3x}\right) = 1$$

derivative of y · integrating factor

so, write in differential form

$$d\left(ye^{-3x}\right) = dx$$

and integrate both sides

$$y e^{-3x} = x + c$$

note: this perfectly acceptable answer is called an implicit solution (not solved for any particular variable)

an explicit solution means that you have solved for the dependent variable

explicit solution here: $y = e^{3x}(x+c)$

Solve the following DE:

$$di + i\,dt = e^{-t}\cos t\;dt$$

$$\frac{di}{dt} + i = e^{-t}\cos t$$

linear with $P(t) = 1$

[form $\frac{dy}{dx} + P(x)\,y = Q(x)$

integrating factor $IF = e^{\int P(t)\,dt} = e^{\int dt} = e^{t}$

$$\frac{di}{dt}e^{t} + i e^{t} = e^{t}e^{-t}\cos t$$

$$\frac{d}{dt}\left(i e^{t}\right) = \cos t$$

$$\int \frac{d}{dt}\left(i e^{t}\right) = \int \cos t\;dt$$

$$\int d(ie^t) = \int \cos t \, dt$$

$$ie^t = \sin t + C$$

Solve the following DE, giving an explicit solution.

$$\frac{dx}{dt} + 2t\,x = t$$

where $x(0) = 1$. linear with $P(t) = 2t$

$$IF = e^{\int P(t)\,dt}$$

$$= e^{\int 2t\,dt}$$

$$= e^{t^2}$$

$$\frac{dx}{dt}e^{t^2} + 2te^{t^2}\cdot x = te^{t^2}$$

$$\frac{d}{dt}\left(xe^{t^2}\right) = te^{t^2}$$

$$\int d\left(xe^{t^2}\right) = \int te^{t^2}dt$$

let $u = t^2$
$du = 2t\,dt$

$$xe^{t^2} = \int \frac{e^u\,du}{2}$$

$$= \frac{1}{2}e^u + C$$

$$xe^{t^2} = \tfrac{1}{2}e^{t^2} + C$$

when $x(0) = 1$ (when $t=0$, $x=1$)

$$1 = \tfrac{1}{2} + C$$
$$C = \tfrac{1}{2}$$

$$x e^{t^2} = \tfrac{1}{2} e^{t^2} + \tfrac{1}{2}$$

$$x = \tfrac{1}{2} + \tfrac{1}{2} e^{-t^2}$$

note: check!

$$x = \tfrac{1}{2} + \tfrac{1}{2} e^{-t^2}$$

$$\frac{dx}{dt} = \frac{1}{2}(-2t)e^{-t^2} = -t e^{-t^2}$$

$$\frac{dx}{dt} + 2tx = t$$

$$-t e^{-t^2} + 2t\left(\frac{1}{2} + \frac{1}{2} e^{-t^2}\right) = t$$

$$\cancel{-t e^{-t^2}} + t + \cancel{t e^{-t^2}} = t \quad \checkmark$$

(Gilles' question!)

$$y\,dx = 4(x + y^6)\,dy$$

is this linear in y? no! y^6 term!

but is it linear in x? divide both sides by dy

$$y \frac{dx}{dy} = 4x + 4y^6$$

$$\frac{dx}{} \quad 4x \quad 4y^5$$

$$\frac{dx}{dy} = \frac{4x}{y} + 4y^5$$

$$\frac{dx}{dy} - \frac{4}{y} x = 4y^5$$

linear with $P(y) = -\frac{4}{y}$

$$IF = e^{\int P(y)\,dy}$$
$$= e^{\int -4/y\,dy}$$
$$= e^{-4\ln y}$$
$$= e^{\ln y^{-4}}$$
$$= y^{-4}$$

$$\frac{dx}{dy} y^{-4} - 4xy^{-5} = 4y^5 \cdot y^{-4}$$

$$\frac{d}{dy}\left(x y^{-4}\right) = 4y$$

$$\int d(xy^{-4}) = \int 4y\,dy$$

$xy^{-4} = 2y^2 + C$ implicit

$x = 2y^6 + Cy^4$ explicit