Section 31.6: Cont'd

Monday, April 15, 2013
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According to Newton's Law of Cooling, the rate at which a body cools (or warms up) is proportional to the difference in temperature between it and the surrounding medium.

a) Write a differential equation expressing this proportionality. (Room temp = T_1)

b) Solve the DE to find the relationship between the temperature of an object as a function of time when it is initially at T_0 and placed in a room at temp T_1.

a)
$$\frac{dT}{dt} \propto T - T_1$$

$$\frac{dT}{dt} = -k(T-T_1)$$

hot things in a cool room will decrease in temperature

b)
$$\frac{dT}{dt} = -k(T-T_1)$$

$$\int \frac{dT}{T-T_1} = \int -k\,dt$$

$$\ln(T-T_1) = -kt + C_1$$

$$T-T_1 = e^{-kt+C_1}$$

$$= e^{-kt}e^{C_1}$$

e^{C_1} = constant

$$T-T_1 = C_2 e^{-kt}$$

at time $t=0$, $T=T_0$

$$T_0 - T_1 = c_2 \cdot 1$$

$$c_2 = T_0 - T_1$$ ← initial temperature difference

$$T - T_1 = (T_0 - T_1)\, e^{-kt}$$

$$T = T_1 + (T_0 - T_1)\, e^{-kt}$$

what happens at $t \to \infty$? $e^{-kt} \to 0$

and $T \to T_1$

Use your DE from the previous example in the following scenario: Beaker goes for a swim in some liquid nitrogen and is frozen at a temperature of 77 K (which is approx −200°c). If Dr Bunsen Honeydew hauls him out into a room at 20°C and it takes Beaker a day to get to −100°C, when will Beaker be at 0°C?

$$T = T_1 + (T_0 - T_1)\, e^{-kt}$$

find k using the info about −100°C:

$$-100 = 20 + (-200 - 20)\, e^{-k \cdot 1}$$

$$-120 = -220\, e^{-k}$$

$$\frac{120}{220} = e^{-k}$$

$$\ln\left(\frac{120}{220}\right) = -k$$

$$k = -\ln\left(\frac{120}{220}\right)$$

$$\approx 0.606136$$

(note: do not round constant in exponents - keep extra digits!)

now, find t:

$$T = T_1 + (T_0 - T_1)\,e^{-kt}$$

$$0 = 20 + (-200 - 20)\,e^{-0.606136t}$$

$$-20 = -220\,e^{-0.606136t}$$

$$\frac{20}{220} = e^{-0.606136t}$$

$$\ln\left(\frac{20}{220}\right) = -0.606136t$$

$$t = \frac{\ln(20/220)}{-0.606136}$$

$$\approx 3.956 \text{ days}$$

$$\approx 4 \text{ days}$$

A tank contains 200 L of fluid in which 100 g of salt is dissolved initially. Brine containing 5 grams of salt per litre is then pumped into the tank at a rate of 4 litres per minute. The well-mixed solution is pumped at at the same rate.

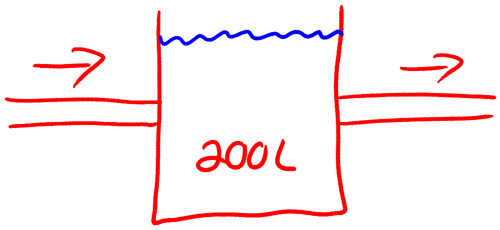
a) find the mass $m(t)$ of salt in the tank as a function of time t.

b) find $\lim_{t \to \infty} m(t)$.

c) after how much time is there 600 g of salt in the tank?

4L/min

5g/L



4 L/min

so concentration of stuff leaving is $\frac{m}{200}$ g/L

initially: 100g in 200L

0.5g/L

m = mass of salt (g)

concentration: $\frac{m}{200} \frac{g}{L}$

rate of brine coming in is:

$$4 \frac{L}{min}$$

rate of salt coming in is:

$$5 \frac{g}{L} \cdot 4 \frac{L}{min} = 20 \frac{g}{min}$$

rate of brine going out is:

$$4 \frac{L}{min}$$

rate of salt going out is:

$$\frac{m}{200}\,\frac{g}{L}\cdot 4\,\frac{L}{min} = \frac{m}{50}\,\frac{g}{min}$$

total rate of change of salt:

$$\frac{dm}{dt} = \left(\frac{dm}{dt}\right)_{incoming} - \left(\frac{dm}{dt}\right)_{outgoing}$$

$$= 20\,\frac{g}{min} - \frac{m}{50}\,\frac{g}{min}$$

writing without units:

$$\frac{dm}{dt} = 20 - \frac{m}{50}$$

$$\frac{dm}{\left(20 - \frac{m}{50}\right)} = dt$$

let $u = 20 - \frac{m}{50}$

$du = -\frac{1}{50}\,dm$

$$\int -50\,\frac{du}{u} = \int dt$$

$$-50 \ln u = t + C_1$$

$$-50 \ln\left(20 - \frac{m}{50}\right) = t + C_1$$

$$\ln\left(20 - \frac{m}{50}\right) = -\frac{t}{50} - \frac{C_1}{50}$$

= constant

$$\ln\left(20 - \frac{m}{50}\right) = -\frac{t}{50} + C_2$$

$$20 - \frac{m}{50} = e^{-t/50 + C_2}$$

$$20 - \frac{m}{50} = e^{-t/50} e^{C_2}$$

constant

$$20 - \frac{m}{50} = C_3 e^{-t/50}$$

$$-\frac{m}{50} = C_3 e^{-t/50} - 20$$

$$m = -50 C_3 e^{-t/50} + 1000$$

$$= 1000 - 50 C_3 e^{-t/50}$$

initial conditions: at $t = 0$, $m = 100$ g

$$100 = 1000 - 50 C_3 \cdot 1$$

$$50 C_3 = 900$$

$$m = 1000 - 900 e^{-t/50}$$

b)

$$\lim_{t \to \infty} m = \lim_{t \to \infty} \left(1000 - \underbrace{900 e^{-t/50}}_{\text{approaches zero}}\right)$$

$$= 1000 \text{ g}$$

note: concentration of salt as $t \to \infty$

is 1000 g . 5 g/l . same as

200 L incoming

c) at what t does $m = 600$?

$$m = 1000 - 900 e^{-t/50}$$

$$600 = 1000 - 900 e^{-t/50}$$

$$-400 = -900 e^{-t/50}$$

$$\frac{4}{9} = e^{-t/50}$$

$$\ln \frac{4}{9} = -\frac{t}{50}$$

$$t = -50 \ln\left(\frac{4}{9}\right)$$

$$\approx 40.5465 \text{ min}$$

$$\approx \left.\begin{array}{l} 40 \text{ min} \\ 41 \text{ min} \\ 40.5 \text{ min} \end{array}\right\} \text{acceptable!}$$