Section 31.7 & 31.8: Cont'd

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1:27 PM

So, how do you come up with the solutions we've been using to 2^{nd}-order linear homogeneous DEs?

start from scratch: (I do not require you to use this formal method)

solve the DE:

$$y'' - 2y' - 8y = 0$$

let's rewrite this:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0$$

and then rewrite again using something called the "operator" form:

replace $\frac{d}{dx}$ by D

$$D^2y - 2Dy - 8y = 0$$

$$(D^2 - 2D - 8)y = 0$$

$$(D-4)(D+2)y = 0$$

let $z = (D+2)y$

then $(D-4)z = 0$

$$\frac{dz}{dx} - 4z = 0$$

$$\frac{dz}{dx} = 4z$$

$$\int \frac{dz}{z} = \int 4\,dx$$

$$\ln z = 4x + c^*$$

$$z = e^{4x + c^*}$$

$$= C e^{4x}$$

substitute back:

$$(D+2)\,y = C e^{4x}$$

$$\frac{dy}{dx} + 2y = C e^{4x} \quad \Leftarrow \text{1st order linear DE}$$

integrating factor: $e^{\int P(x)dx} = e^{\int 2dx} = e^{2x}$

$$\frac{dy}{dx} e^{2x} + 2y e^{2x} = C e^{4x} e^{2x}$$

$$\frac{d}{dx}\left(y e^{2x} \right) = C e^{6x}$$

$$\int d\left(y e^{2x}\right) = \int C e^{6x}\,dx$$

$$y e^{2x} = \underbrace{\frac{1}{6} C}_{= C_1} e^{6x} + C_2$$

$$y e^{2x} = C_1 e^{6x} + C_2$$

$$y = C_1 e^{4x} + C_2 e^{-2x}$$

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what about the case of one repeated real solution?

solve

$$y'' - 8y' + 16y = 0$$

$$D^2 y - 8Dy + 16y = 0$$

$$(D^2 - 8D + 16)y = 0$$

$$(D-4)(D-4)y = 0$$

let $z = (D-4)y$ ⟸ note: we just solved this!

$$z = Ce^{4x}$$

so, substituting back,

$$(D-4)y = Ce^{4x}$$

$$\frac{dy}{dx} - 4y = Ce^{4x}$$

integrating factor $e^{\int P(x)dx}$

$$= e^{\int -4\,dx}$$

$$= e^{-4x}$$

$$\frac{dy}{dx}e^{-4x} - 4ye^{-4x} = Ce^{4x}e^{-4x}$$

$$\frac{d}{dx}\left(ye^{-4x}\right) = C$$

$$\int d\left(ye^{-4x}\right) = \int C\,dx$$

$$ye^{-4x} = Cx + C_2$$

$$y = (Cx + C_2)e^{4x}$$

examples:

solve the following DEs:

$$9\frac{d^2y}{dt^2} + 4y = 0$$

auxiliary equation: $9m^2 + 4 = 0$

$$m^2 = -\frac{4}{9}$$

$$m = \pm \frac{2}{3} i$$

so $m = \alpha \pm \beta i$ where $\alpha = 0$, $\beta = 2/3$

$$y = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$$

$$= C_1 \cos\left(\frac{2}{3}t\right) + C_2 \sin\left(\frac{2}{3}t\right)$$

note: the DE

$$9\frac{d^2y}{dt^2} - 4y = 0$$

has solutions: $9m^2 - 4 = 0$

$$m^2 = \frac{4}{9}$$

$$m = \pm \frac{2}{3}$$

$$y = C_1 e^{2/3 t} + C_2 e^{-2/3 t}$$