section 31.9: cont'd

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1:30 PM

example: solve:

$$y'' + 4y = 2\sin 3x$$

complementary solution y_c (soln to homogeneous case):

aux eqn: $m^2 + 4 = 0$

$m = \pm 2i$

$m = \alpha \pm \beta i$ where $\alpha = 0$

$\beta = 2$

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

$$= C_1 \cos 2x + C_2 \sin 2x$$

particular solution y_p:

$$y_p = A \sin 3x + B \cos 3x$$

$$y_p' = 3A \cos 3x - 3B \sin 3x$$

$$y_p'' = -9A \sin 3x - 9B \cos 3x$$

$$y'' + 4y = 2 \sin 3x$$

$$-9A \sin 3x - 9B \cos 3x + 4A \sin 3x + 4B \cos 3x = 2 \sin 3x$$

$$-5A \sin 3x - 5B \cos 3x = 2 \sin 3x$$

for this equation to be true for all values of x,

then: $-5A = 2$

$A = -\frac{2}{5}$

$$-5B = 0$$
$$B = 0$$

so, the full solution is:

$$y = y_c + y_p$$
$$= C_1 \sin 2x + C_2 \cos 2x - \frac{2}{5} \sin 3x$$

Solve:

$$2y'' + 5y' - 3y = e^x + 4e^{2x}$$

complementary soln:

aux eqn: $2m^2 + 5m - 3 = 0$ $\quad$ $ac = -6$ $\quad$ $-1 \; 6$

$$2m^2 + 6m - m - 3 = 0$$
$$2m(m+3) - 1(m+3) = 0$$
$$(2m-1)(m+3) = 0$$
$$m = -3, \tfrac{1}{2}$$

$$y_c = C_1 e^{-3x} + C_2 e^{\frac{1}{2}x}$$

particular soln:

$$y_p = Ae^x + Be^{2x}$$
$$y_p' = Ae^x + 2Be^{2x}$$
$$y_p'' = Ae^x + 4Be^{2x}$$

$$2y'' + 5y' - 3y = e^x + 4e^{2x}$$

$$2Ae^x + 8Be^{2x} + 5Ae^x + 10Be^{2x} - 3Ae^x - 3Be^{2x} = e^x + 4e^{2x}$$

$$4Ae^x + 15Be^{2x} = e^x + 4e^{2x}$$

$\therefore \quad 4A = 1 \qquad$ and $\qquad 15B = 4$

$A = 1/4 \qquad\qquad B = 4/15$

$$y_p = \tfrac{1}{4} e^{x} + \tfrac{4}{15} e^{2x}$$

so, full solution is:

$$y = y_c + y_p$$
$$= C_1 e^{-3x} + C_2 e^{\frac{1}{2}x} + \tfrac{1}{4} e^{x} + \tfrac{4}{15} e^{2x}$$

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solve:

$$y'' + 9y = 8e^{x}$$

complementary:

$$m^2 + 9 = 0$$
$$m = \pm 3i$$

$$y_c = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$$
$$= C_1 \cos 3x + C_2 \sin 3x$$

particular:

$$y_p = Ae^{x}$$

$$y_p' = Ae^{x}$$

$$y_p'' = Ae^{x}$$

$$y'' + 9y = 8e^{x}$$

$$Ae^{x} + 9Ae^{x} = 8e^{x}$$

$$10Ae^{x} = 8e^{x}$$

$$10A = 8$$

$$A = 4/5$$

$$y_p = \frac{4}{5} e^x$$

$$y = y_c + y_p$$

$$= C_1 \cos 3x + C_2 \sin 3x + \frac{4}{5} e^x$$

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the "bad case":

example: solve $\quad y'' + 9y = 4 \sin 3x$

complementary: aux eqn: $m^2 + 9 = 0$

$m = \pm 3i$

$$y_c = C_1 \cos 3x + C_2 \sin 3x$$

why is this particular example so problematic?

naively, using our previous rules,

$$y_p = A \underbrace{\sin 3x} + B \underbrace{\cos 3x}$$

same functional form as y_c

so, when you have like terms in y_p and y_c, you have "the bad case"

here's what you do: if y_p has terms that look like y_c, then multiply each term by x until y_p has a different form in this example, we'd set

$$y_p = A\,x\sin 3x + B\,x\cos 3x$$
$$= x(A\sin 3x + B\cos 3x)$$

so

$$y_p' = A\sin 3x + B\cos 3x + 3Ax\cos 3x - 3Bx\sin 3x$$
$$= A\sin 3x + B\cos 3x + x(3A\cos 3x - 3B\sin 3x)$$

$$y_p'' = 3A\cos 3x - 3B\sin 3x + 3A\cos 3x - 3B\sin 3x + x(-9A\sin 3x - 9B\cos 3x)$$
$$= 6A\cos 3x - 6B\sin 3x - 9x(A\sin 3x + B\cos 3x)$$

and substitute back in to DE:

$$y'' + 9y = 4\sin 3x$$

$$6A\cos 3x - 6B\sin 3x - \cancel{9x(A\sin 3x + B\cos 3x)} + \cancel{9x(A\sin 3x + B\cos 3x)} = 4\sin 3x$$

$$6A\cos 3x - 6B\sin 3x = 4\sin 3x$$

so $6A = 0$, $A = 0$ and $-6B = 4$, $B = -2/3$

$$y_p = -\frac{2}{3}x\cos 3x$$

full solution:

$$y = y_c + y_p$$
$$= C_1\cos 3x + C_2\sin 3x - \frac{2}{3}x\cos 3x$$

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example: solve:

$$y'' - 2y' + y = xe^{2x} - e^{2x}$$

given that $y' = 4$ and $y = -2$ when $x = 0$

complementary solution: $m^2 - 2m + 1 = 0$

$m = 1$

$$y_c = (c_1 + c_2 x) e^x$$

particular solution: not bad case

$$y_p = Axe^{2x} + Be^{2x}$$

$$y_p' = 2Axe^{2x} + Ae^{2x} + 2Be^{2x}$$

$$y_p'' = 4Axe^{2x} + 2Ae^{2x} + 2Ae^{2x} + 4Be^{2x}$$

$$= 4Axe^{2x} + 4Ae^{2x} + 4Be^{2x}$$

now, substitute back into DE:

$$y'' - 2y' + y = xe^{2x} - e^{2x}$$

$$(\cancel{4Axe^{2x}} + 4Ae^{2x} + 4Be^{2x}) - \cancel{4Axe^{2x}} - 2Ae^{2x} - 4Be^{2x} + Axe^{2x} + Be^{2x} = xe^{2x} - e^{2x}$$

$$Axe^{2x} + 2Ae^{2x} + Be^{2x} = xe^{2x} - e^{2x}$$

$$Axe^{2x} + e^{2x}(2A + B) = xe^{2x} - e^{2x}$$

so $A = 1$ and $2A + B = -1$

$B = -3$

so $y_p = xe^{2x} - 3e^{2x}$

and

$$y = y_c + y_p$$

$$= (c_1 + c_2 x)e^x + xe^{2x} - 3e^{2x}$$

so, initial conditions:

when $x=0$, $y=-2$

$$-2 = C_1 - 3$$

so $C_1 = 1$

when $x=0$, $y'=4$

$$y' = (C_1 + C_2 x)e^x + C_2 e^x + 2xe^{2x} + e^{2x} - 6e^{2x}$$

$$4 = C_1 + C_2 + 1 - 6$$

$$4 = 1 + C_2 + 1 - 6$$

$$C_2 = 8$$

$$y = (1+8x)e^x + (x-3)e^{2x}$$