Section 31.10: cont'd

Monday, April 29, 2013
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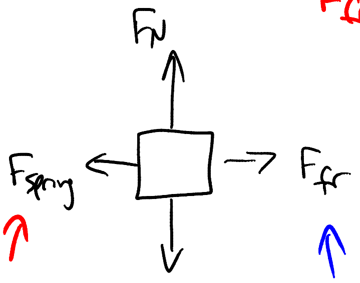
what if, for example, there is air resistance?

$$F_{fr} \propto v$$

$$F_{fr} = bv$$



direction depends on x

direction depends on velocity

$$\Sigma \vec{F} = m\vec{a}$$

$$F_{spring} - F_{fr} = ma$$

$$-kx - bv = ma$$

$$0 = ma + bv + kx$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$(m\ddot{x} + b\dot{x} + kx = 0)$$

2nd order linear homogeneous:

aux eqn: $\quad m\,n^2 + bn + k = 0$

$$n = \frac{-b \pm \sqrt{b^2 - 4km}}{2m}$$

solutions will be:

① 2 real if $b^2 - 4km > 0$
② 1 real $b^2 - 4km = 0$
③ 2 complex $b^2 - 4km < 0$

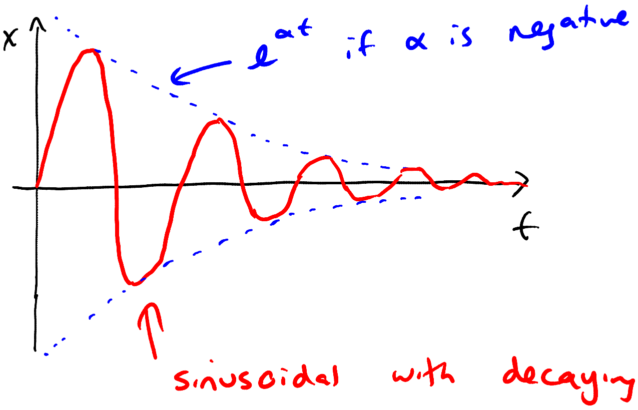
①: $x_1 = c_1 e^{n_1 t} + c_2 e^{n_2 t}$

②: $x_2 = (c_1 + c_2 t) e^{nt}$

③: $x_3 = e^{\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t)$

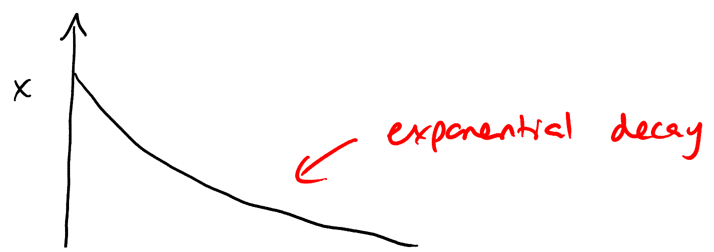
what do these solutions look like?

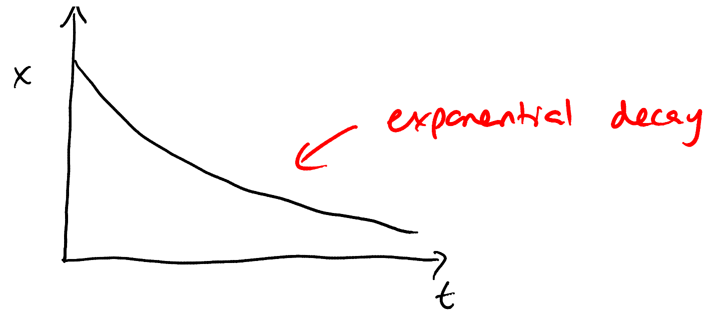
③



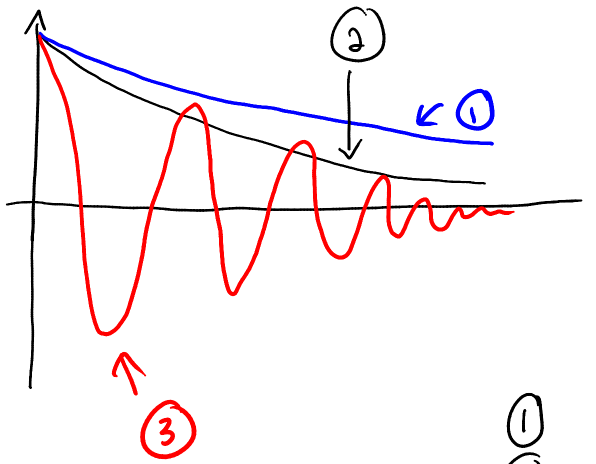
but what do ① and ② look like?

if the n's are negative (sol'ns to aux eqn)





so: putting it all together:



1 overdamped
2 critically damped
3 underdamped

critically damped - just enough friction to prevent oscillation

→ object "returns to equilibrium" in minimum time

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so, what about an external force?

then get

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = \underbrace{F_{ext}(t)}_{\text{so DE is now}}$$

non
inhomogeneous

this is the most general case

suppose we have no friction $(b=0)$ but an external force

$$F_{ext}(t) = F_0 \cos \omega t$$

so our DE is now:

$$m \frac{d^2x}{dt^2} + kx = F_0 \cos \omega t$$

complementary sol'n:

aux eqn: $mn^2 + k = 0$

$n^2 = -\frac{k}{m}$

$\alpha = 0$, $\beta = \sqrt{\frac{k}{m}}$ $\longrightarrow$ $n = \pm \sqrt{\frac{k}{m}}\, i$

$$x_c = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

$$= C_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

$$= C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

where ω_0 = natural frequency $= \sqrt{\frac{k}{m}}$

particular solution:

$$m \frac{d^2x}{} + kx = F_0 \cos \omega t$$

$\overline{dt^2}$

look at RHS:

$$x_p = A\cos\omega t + B\sin\omega t \qquad \omega \neq \omega_0$$

how do we find A and B? take derivatives and plug into DE

$$\frac{dx_p}{dt} = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$\frac{d^2x_p}{dt^2} = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$$

$$= -\omega^2(A\cos\omega t + B\sin\omega t)$$

DE:

$$m\frac{d^2x}{dt^2} + kx = F_0\cos\omega t$$

$$-m\omega^2(A\cos\omega t + B\sin\omega t) + k(A\cos\omega t + B\sin\omega t) = F_0\cos\omega t$$

$$(-m\omega^2 + k)A\cos\omega t + (-m\omega^2 + k)B\sin\omega t = F_0\cos\omega t$$

so $(-m\omega^2 + k)A = F_0$

$$A = \frac{F_0}{-m\omega^2 + k}$$

and $(-m\omega^2 + k)B = 0$

$$B = 0$$

$x_p = F_0 \quad \cos\omega t$

$$-m\omega^2+k$$

and $$x = x_c + x_p$$

$$= C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{k-m\omega^2} \cos \omega t$$

note that $\omega_0 = \sqrt{\frac{k}{m}}$ so $\omega_0^2 = \frac{k}{m}$

and $k = m\omega_0^2$

so the particular solution can be written

$$x_p = \frac{F_0}{m(\omega_0^2-\omega^2)} \cos \omega t$$

so what happens when ω approaches ω_0 (the frequency with which you are pumping your oscillator is close to the natural frequency of that oscillator)?

denominator goes to zero!

amplitude of oscillation goes to infinity!

=> resonance (pumping something at its natural frequency)

note: pump with $\omega < \omega_0$, mass moves in phase with your hand

$\omega > \omega_0$, out of phase

$\omega \approx \omega_0$, resonance

can get resonance with earthquakes acting on buildings and other structures

older cars driven at right speed on highways

etc

note: to reduce effects of vibrations, introduce friction (damping)

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example:

consider a mass on a spring on a frictionless surface, but do not neglect air resistance.

a) write the differential equation if there are no external forces acting on the mass

b) if $m = 5$ kg and $k = 30$ N/m, what value of b leads to critical damping?

a) $$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

where m, b, and k are positive constants

b) aux eqn: $mn^2 + bn + k = 0$

$$n = \frac{-b \pm \sqrt{b^2 - 4km}}{2m}$$

critically damped : 1 real solution

so $b^2 - 4km = 0$

$$b^2 = 4km$$

$$b = \pm\sqrt{4km}$$

$$= \sqrt{4km}$$

$$= \sqrt{4 \; 30 \; 5}$$

$$= \sqrt{600}$$

$$= 10\sqrt{6}$$

$$\approx 24.5$$