Math 189 – Test #1

April 30, 2015 **Name:** Solution Set
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Total: 25 points

1. Consider the solution $y = c^2 + \frac{c}{x}$ to the differential equation $x^4 (y')^2 - xy' = y$.

(4 points)

a) State the order and degree of the DE. first order, second degree (1)

b) Is this solution a general or particular solution? general (1)

c) Show that the given solution really is a solution to the DE.

$$y = c^2 + \frac{c}{x}$$

$$y' = -\frac{c}{x^2} \quad (1)$$

sub into DE.

$$x^4 (y')^2 - xy' = y$$

$$x^4 \left(-\frac{c}{x^2}\right)^2 - x\left(-\frac{c}{x^2}\right) = c^2 + \frac{c}{x}$$

$$x^4 \frac{c^2}{x^4} + \frac{xc}{x^2} = c^2 + \frac{c}{x}$$

$$c^2 + \frac{c}{x} = c^2 + \frac{c}{x}$$

(1)

✓

2. Solve the <u>linear</u> differential equation, subject to the given initial condition. Give an explicit solution. (6 points)

$$3x\,dy - y\,dx = 9x\,dx$$

where $y(1) = 3$

$$\frac{dy}{dx} - \frac{1}{3x}y = 3$$

1

linear with $P(x) = -\frac{1}{3x}$ and integrating factor

$$e^{\int P(x)dx} = e^{\int -1/3x\,dx}$$ 1

$$= e^{-1/3 \ln x} = e^{\ln x^{-1/3}} = x^{-1/3}$$ 1

$$x^{-1/3}\frac{dy}{dx} - \frac{x^{-4/3}}{3}y = 3x^{-1/3}$$ 1

$$\frac{d}{dx}\left(x^{-1/3}y\right) = 3x^{-1/3}$$

$$\int d\left(x^{-1/3}y\right) = \int 3x^{-1/3}\,dx$$

$$x^{-1/3}y = \frac{3x^{2/3}}{2/3} + C$$

$$y = 9/2\,x + Cx^{1/3}$$ 1

when $x = 1, y = 3$:

$$3 = 9/2 \cdot 1 + C$$

$$C = 3 - 9/2 = -3/2$$

so $\boxed{y = 9/2\,x - 3/2\,x^{1/3}}$ 1

3. Solve: $y'' + 8y' + 25y = 0$. (3 points)

$m^2 + 8m + 25 = 0$ (½)

$m = \dfrac{-8 \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-8 \pm \sqrt{64 - 100}}{2} =$

$= \dfrac{-8 \pm \sqrt{-36}}{2}$

$= \dfrac{-8 \pm 6i}{2} = -4 \pm 3i = \alpha \pm \beta i$ where $\alpha = -2$ and $\beta = 3$ (1)

$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$ (½)

$\boxed{y = e^{-4x}(C_1 \cos 3x + C_2 \sin 3x)}$ (1)

4. Consider the differential equation $\dfrac{dy}{dx} = f(x, y)$. For the following questions about numerical methods, circle the correct answers. You may choose more than one. (2 points)

When you use Euler's method to approximate the solution to the DE, the solution is

a) A point
b) A formula for y as a function of x
c) A table of values

(1)

Why is the Improved Euler method (also known as Heun's method) a more accurate method for approximating a DE than the Euler method?

d) The Improved Euler method uses a smaller step size.
e) The Improved Euler method uses more decimal points.
f) The Euler method is a "predictor" method, while Improved Euler is a "predictor-corrector" method.
g) The Euler method uses the slope at the left-hand side of the interval to predict the y-value at the right-hand side of the interval, while the Improved Euler method uses the average of the slope at the left-hand side and an estimate of the slope on the right for its prediction.

(1)

5. Solve the following differential equation. (4 points)

$$x\,dt + t\,dx - \csc(xt)dt = 0$$

let $u = xt$

$du = t\,dx + x\,dt$ 1

$du - \csc u\,dt = 0$

$du = \csc u\,dt$

$\frac{du}{\csc u} = dt$ 1

← if don't separate, so get stuck here −3

$\int \sin u\,du = \int dt$

$-\cos u = t + C$ 1

$\boxed{-\cos xt = t + C}$ 1

← no +C −1

or $\boxed{t + \cos xt = C_1}$ etc

forget to substitute back −1

6. In his latest attempt to catch the roadrunner, the wily coyote has ordered some rocket-powered rollerskates from ACME Corp. To test the skates out, he straps them on and, starting from rest, fires the rockets. After ignition, the coyote finds that the acceleration of the skates (<u>and</u> the unhappy coyote) towards the nearest cliff is proportional to the square root of the velocity, and that after exactly 3 seconds he has reached a velocity of 144 m/s. (6 points)

   a) Write an equation showing the relationship between the coyote's acceleration and his velocity. Then change your equation into a DE using the fact that the acceleration is the rate of change of the velocity.
   b) Solve the DE in part (a) to find an expression for the coyote's velocity as a function of the time elapsed after rocket ignition.
   c) At what time is the coyote zooming towards the cliff at 100 m/s?

a) $a \propto \sqrt{v}$ so $a = k\sqrt{v}$

$$\boxed{\frac{dv}{dt} = k\sqrt{v}}$$ 1

b) $$\frac{dv}{\sqrt{v}} = k\,dt$$ separable 1

$$\int v^{-1/2}\,dv = \int k\,dt$$

$$2\sqrt{v} = kt + C$$ 1

at $t=0, v=0$ (starting from rest), so $0 = 0 + C$
$C = 0$ 1

at $t=3, v=144$, so $2 \cdot \sqrt{144} = k \cdot 3$
$24 = 3k$
$k = 8$

and $2\sqrt{v} = 8t$
$\sqrt{v} = 4t$
$$\boxed{v = 16t^2}$$ 1

c) $v = 100$, so
$100 = 16t^2$
$t^2 = \frac{100}{16} = \frac{25}{4}$

$$\boxed{t = 2.5 \text{ s}}$$ 1